

## **Wavefield extrapolation by windowed nonstationary phase shift**

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### **ABSTRACT**

The nonstationary phase-shift extrapolator provides a highly accurate wavefield extrapolator in laterally inhomogeneous media. However, because the operator depends on both the lateral spatial variable and its wavenumber, the transform between space and wavenumber cannot be implemented with the fast Fourier transform technique and therefore, the computational effort is greater than that of the stationary phase shift method. In this paper, a method is developed that applies the windowed Fourier transform to the Helmholtz equation to obtain a local wavefield expression, which leads to the windowed version of nonstationary phase-shift. Numerical results show that the computational efficiency is improved with little loss of accuracy.

### **INTRODUCTION**

Implementation of wavefield extrapolation in the frequency-wavenumber domain has many advantages, e.g., the spatial derivatives are calculated accurately with the Fourier transform (Orsag, 1972) and ray-tracing is not necessary (Goran and Richards, 1991). Wavefield extrapolation is mostly based on the one-way wave equation, which comes from the factorization of Helmholtz equation. The exact factoring of the Helmholtz equation for the apparently simple case of a homogeneous medium (constant velocity), results in a nonlocal one-way wave equation. The resulting phase-shift method (Gazdag, 1978) is an exact solution for the extrapolated one-way wavefield. The phase-shift method is attractive because the computation is very efficient due to its use of the fast Fourier transform (FFT) for all integrations. However, the requirement of constant velocity limits this method because seismic imaging must often cope with laterally inhomogeneities.

Recently, Margrave and Ferguson (1999) presented a nonstationary phase-shift method for wavefield extrapolation based on pseudo-differential operator theory. With this operator, the Helmholtz wave equation can be approximately factorized into a one-way wave equation with high accuracy for the case of velocity varying laterally. The operator can be applied in either the frequency-wavenumber domain or the frequency-space domain. There are two forms of nonstationary phase shift algorithms, i.e. the combination extrapolator (PSPI in the nonstationary limit) and the convolution extrapolator (nonstationary phase shift, NSPS). Margrave and Ferguson (1998) recognize that PSPI and NSPS are the transpose of one another in the space-frequency domain and can be combined to produce forms that are symmetric. Symmetry of explicit extrapolators in space domain is required by reciprocity considerations (Wapenaar and Grimbergen, 1996).

Unlike the stationary operator that is a function either of the position vector (space domain) or of wavenumber (wavenumber domain), the nonstationary operator is the function of both variables. In this case, the FFT cannot be applied to the transform between the space and wavenumber domain so extra computational effort is required for nonstationary phase shift extrapolation.

When the variation of velocity is weak, several less accurate methods can be applied with satisfactory results. For example, in split-step-Fourier method (Stoffa, et al., 1990) and phase-screen method (Wu, 1998), the operator can be approximately split into two independent operators and each of them is only a function of spatial variable or wavenumber, respectively. Therefore, the fast Fourier transform can be utilized to obtain computational efficiency. However, this method fails in the case of strong velocity variation (Stoffa, 1990).

In this paper, a windowed Fourier transform technique is applied to the Helmholtz wave equation. Applying the split-step technique to the solutions of nonstationary phase-shift leads to the three split-step Fourier schemes. The advantage of this windowed version of nonstationary phase shift is that the FFT can be applied to obtain the computational efficiency and accuracy.

### THE WINDOWED FOURIER TRANSFORM

Let  $W(t)$  be a function that vanishes outside the interval  $-L \leq t \leq 0$ , i.e., such that  $W(t) \subset [-L, 0]$ .  $W(t)$  will be weight function, or window, which will be used to localize the function  $f(t)$  in time. For a specific time  $t$ , define

$$f_{\chi}(t) = W(t - \chi)f(t) \tag{1}$$

then,  $f_{\chi}(t) \subset [\chi-L, \chi]$ , and we think of  $f_{\chi}(t)$  as a localized version of  $f(t)$  that depends only on the values of  $f(t)$  for  $\chi-L \leq u \leq \chi$ . If  $W$  is continuous, then  $f_{\chi}(t)$  with  $t \approx \chi-L$  and  $t \approx \chi$  are small. This means that the above localization is smooth rather than abrupt. The windowed Fourier transform of  $f$ , or the Fourier transform of  $f_{\chi}$  is (Kaiser, 1994, A friendly guide to wavelets, Birkhauser)

$$\tilde{f}(\chi, \omega) = \tilde{f}_{\chi}(\omega) = \int W(\chi - t)f(t)e^{-i\omega t} dt \tag{2}$$

In equation (1), the output of the Fourier transform  $\tilde{f}_{\chi}(\omega)$  depends on  $f(t)$  only for the values inside the window and gives little weight to the values of  $f(t)$  near the endpoints. The windowed Fourier transform is a real-time replacement for the Fourier transform, giving the dynamical (time-varying) frequency distribution of  $f(t)$ , which allows us to analyze the spectrum of  $f$  in a local time window. In the limiting case when  $W \equiv 1$  the windowed Fourier transform reduces to the ordinary Fourier transform.

The corresponding inverse Fourier transform, i.e. to reconstruct  $f$  from  $\tilde{f}$ , can be performed by applying the inverse Fourier transform, with respect to the variable  $\omega$  to equation (2), we obtain

$$W(t - \chi)f(t) = \int \tilde{f}_\chi(\omega) \exp(i\omega t) d\omega \quad (3)$$

In equation (3) we cannot recover  $f(t)$  by dividing by  $W(t-\chi)$  since this function may vanish. Instead, we multiply equation (2) by  $W^*(t-\chi)$  and integrate over  $\chi$ , i.e.

$$\int |W(t - \chi)|^2 f(t) d\chi = \int d\chi \int d\omega \exp(i\omega t) W^*(t - \chi) \tilde{f}(\omega, \chi) \quad (4)$$

The left-hand side is simply equal to  $\|W\|^2 f(t)$ , hence

$$f(t) = C^{-1} \int d\chi \int d\omega \exp(i\omega t) W(t - \chi) \tilde{f}(\omega, \chi) \quad (5)$$

where \* denotes the complex transpose and  $C = \|W\|^2$ .

## LOCAL FOURIER TRANSFORMED HELMHOLTZ WAVE EQUATION AND THE SOLUTIONS

The wave equation in the temporal frequency domain is the Helmholtz equation

$$\left( \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial x^2} + \omega^2 s^2(x) \right) \psi(x, z; \omega) = 0 \quad (6)$$

where,  $\psi(x, z; \omega)$  is the wavefield in two spatial coordinates,  $\omega$  is the angular frequency, and  $s(x)$  is the slowness that depends on the horizontal coordinate  $x$ . If the local spectrum is defined as

$$\varphi(\chi, k_x, z, \omega) = \int W(x - \chi) \psi(x, z, \omega) e^{-ik_x x} dx \quad (7)$$

then the equation (6) can be written as

$$\int dx W(\chi - x) e^{-ik_x x} \nabla^2 \psi + \int dx W(\chi - x) e^{-ik_x x} \omega^2 s^2 \psi = 0 \quad (8)$$

Following the derivation given by Hildebrand (1987), equation (8) can be expressed as

$$\left\{ \frac{d^2}{dz^2} + \omega^2 \bar{s}(\chi, z) - \bar{k}_x^2 \right\} \varphi(\chi, k_x, z; \omega) \quad (9)$$

where

$$\bar{s}^2(\chi, z) = s_0^2(\chi, z) + \Delta s^2(x - \chi) \quad (10a)$$

where  $s_0$  is the reference slowness and  $\Delta s^2 = (s^2 - s_0^2)$ , and

$$\bar{k}_x = k_x + i \frac{d}{dx} \ln(W) \quad (10b)$$

Equation (9) is very similar to the normal Helmholtz equation if the variable  $\chi$  is dropped. Based on the nonstationary phase-shift, the approximate solution for wavefield extrapolation can be written as

$$\psi = \int \alpha(k_x, x, \Delta z) \phi \exp(ik_x x) dk_x \quad (11)$$

and

$$\phi = \int \alpha(k_x, x, \Delta z) \psi \exp(-ik_x x) dx \quad (12)$$

which are the PSPI and NSPS versions, respectively. In the equations above,  $\alpha$  is the nonstationary extrapolator defined as

$$\alpha(k_x, x, \Delta z) = \exp(\pm i \sqrt{\omega^2 \bar{s}^2 - k_x^2} \Delta z) \quad (13)$$

The extrapolator  $\alpha$  can be considered as the generalized phase shift when the wavefield is propagating up and down along the  $z$ -axis (Wenzel, 1991). By combining equations (11) and (12), a symmetric form of solution can be obtained (Margrave and Ferguson, 1999)

$$\psi(x, z + \Delta z) = \frac{1}{2} \int \psi(x_1, z) \int [\alpha(k_x, x, \Delta z) + \alpha(k_x, x_1, \Delta z)] \exp(-k_x \cdot (x - x_1)) dk_x dx_1 \quad (14)$$

So far, no improvement has been made in the computational efficiency, because the extrapolator is still a function of both wavenumber and  $x$ . If we write

$$\sqrt{\omega^2 \bar{s}^2 - k_x^2} = \sqrt{\omega^2 s_0^2 - k_x^2 + \omega^2 \Delta s^2}$$

and choose a proper window so that the slowness variation within the window is small. Then the square root can be expressed as

$$\sqrt{\omega^2 \bar{s}^2 - k_x^2} \approx k_x + \frac{\omega^2}{2k_x} \Delta s^2 \quad (15)$$

where

$$k_z = \sqrt{\omega^2 s_0^2 - k_x^2} \quad (16)$$

Moreover, if the depth-step for the extrapolation is sufficiently small, the energy of the wavefield is locally concentrated. Therefore, the energy propagation is then mostly vertical and the second term on the right of equation (15) can be approximated as  $\frac{\omega \Delta s^2}{2s_0}$ . Then, the extrapolator in equation (13) can be approximated as

$$\alpha(k_x, x, \Delta z) = \exp(A + B)\Delta z \quad (17)$$

where

$$A = ik_z \quad B = i \frac{\omega \Delta s^2}{2s_0} \quad (18)$$

Substituting equation (17) into (11) and (12) and using the technique of split-step (Hardin and Tappert, 1973) the windowed version solution of nonstationary phase-shift can be written as

$$\psi(z + \Delta z, x; \omega) = F_x^{-1}[e^{A\Delta z} e^{B\Delta z} F_x(\psi(x, z; \omega))](PSPI) \quad (19)$$

and

$$\psi(x, z + \Delta z; \omega) = F_x^{-1}[F[e^{A\Delta z} e^{B\Delta z} \psi(x, z; \omega)]](NSPS) \quad (20)$$

Equations (19) and (20) are equivalent to the split-step Fourier method except that the solutions are functions of the window parameter  $\chi$ . The computational cost is reduced because FFT can now be used. This is also the method implemented in Ferguson and Margrave (1999) when  $W$  is a boxcar function and within the boxcar the velocity variation can be ignored.

The windowed form corresponding to equations (19) and (20) can be written as

$$\psi(x, z + \Delta z; \omega) = e^{B\Delta z} F_x^{-1}[e^{A\Delta z} F_x(\hat{\psi}(x, z; \omega))] \quad (21)$$

and

$$\psi(x, z + \Delta z; \omega) = F_x^{-1}[e^{A\Delta z} F[e^{B\Delta z} \psi(x, z; \omega)]] \quad (22)$$

Combing (21) and (22) with a half step in  $\Delta z$ , we can obtain

$$\psi(x, z + \Delta z; \omega) = e^{B\Delta z/2} F_x^{-1} [e^{A\Delta z} F_x [e^{B\Delta z/2} \psi(x, z; \omega)]] \quad (23)$$

which corresponds to the symmetric solution of nonstationary phase-shift described by equation (14).

### RECONSTRUCTION OF THE WAVEFIELD

The solutions given by equations (21-23) are windowed versions that depend on the local spatial variable. Therefore, they must be transformed back to the global spatial variable to reconstruct the wavefield. The reconstruction can be implemented via equation (5). For example, if the window is chosen as a Gaussian function (Kaiser, 1994),

$$W(x) = \left(\frac{a}{\pi}\right)^{1/4} \exp(-\pi a x^2) \quad (24)$$

where  $a$  gives the width of the window. When  $a=1$ , it reduces to the Gabor transform. The reconstruction can be obtained by,

$$\psi(x, z + \Delta z; \omega) = \frac{1}{2\pi} \int W^*(x - \chi) \psi(\chi, x, z + \Delta z; \omega) d\chi \quad (25)$$

In practice, when velocity variation is not strong, a simpler window function may be designed as

$$W(x - \chi) = 1 \quad |x - \chi| < a \quad (26a)$$

$$W(x - \chi) = \exp\{-\eta(x - \chi - a)^2\} \quad |x - \chi| > a \quad (26b)$$

Equation (26) designs a tapered boxcar function and  $\eta$  is a constant that controls the taper length. The reconstruction can be performed by (Hildebrand, 1987)

$$\psi(x, z + \Delta z; \omega) = \frac{1}{2\pi} \int C(x - \chi) \psi(\chi, x, z + \Delta z; \omega) d\chi \quad (27)$$

where  $C(x-\chi)=1$  when  $|x-\chi|<a$  and otherwise is zeros.

After wavefield reconstruction for a single window, only the portion with width “ $a$ ” at the centre of the window is used. The final extrapolated wavefield at the depth  $z+\Delta z$  can then be obtained by combining all of the windowed solution. When the window function is taken as a  $\delta$ -function, then the solution given by original nonstationary phase-shift is obtained.

## NUMERICAL EXAMPLES

The first model used for simulation is shown in figure 1. The model is composed of two blocks. The velocity on the left block is 1500 m/s and on the right it is 2500 m/s. One point source is located close to the velocity boundary on the left. Two windows are used in the simulation.

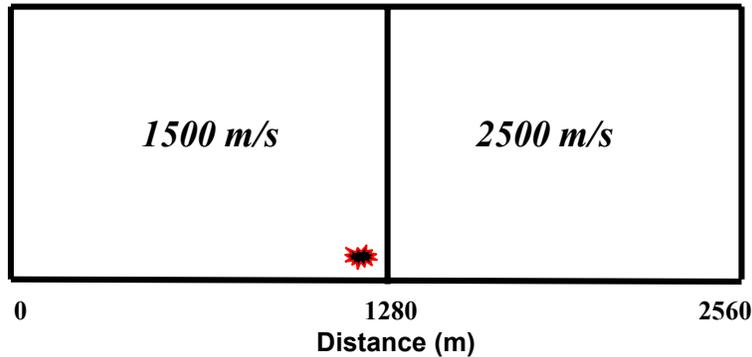


Fig. 1. Velocity model for Model I.

After extrapolation with  $z=200\text{ m}$  (20 steps of  $\Delta z=10\text{ m}$ ) the result is shown in Figure 2a. The traveltimes of the first arrival calculated by solving the Eikonal equation with finite-difference method (Vidale, 1988) matches our result very well. Figure 2b shows the result when the wavefield is extrapolated back (20 steps of  $\Delta z=-10\text{ m}$ ) to  $z=0$ . The original spike pulse is well recovered, which means that the method is consistent for both forward and backward extrapolations.

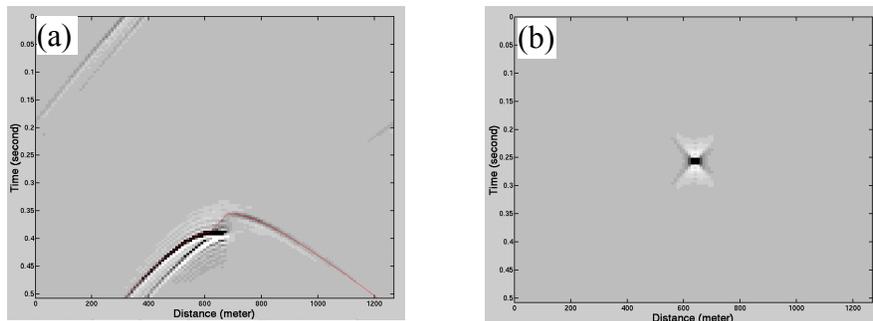


Figure 2. The result of (a), the wavefield extrapolated with  $z=200\text{ m}$  and (b), with  $z=-200\text{ m}$  with the data from experience (a).

These results are very similar to the results produced by nonstationary phase-shift (Yao and Margrave, 1999), because essentially both methods are the same for such a model.

The second model is shown in Figure 3 for the post stack migration test. The exploding reflector shown in Figure 3a is used for modeling the wave propagation in the velocity model shown in Figure 3b. The synthetic data (Figure 4) was generated

by the Fourth-order staggered grid finite-difference method. In Figure 4, the data in the left part contains coherent noise (head waves off the vertical interface) because the velocity there is much lower than that of the right part. There is also more dispersion on the left. However, the existence of these noises can also be used to test the flexibility of the method.

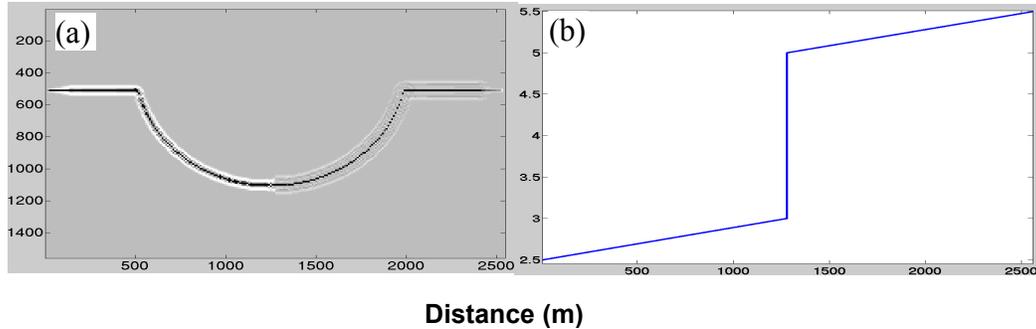


Fig. 3. Model II. (a) exploding reflector and (b) velocity model.

The result shown in Figure 5a is produced by the windowed nonstationary phase-shift with two windows that covers left and right part, respectively. The image is produced by each 20m extrapolating step. Compared with the more accurate result (Figure 5b) produced by the eigenfunction decomposition method (Yao and Margrave, 1999) shows that the simplified windowed nonstationary extrapolator works very well.

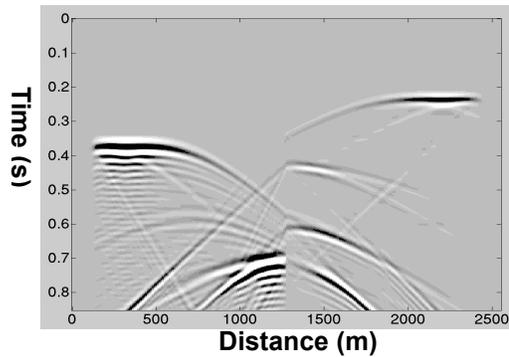


Fig. 4. Synthetic data from Model II

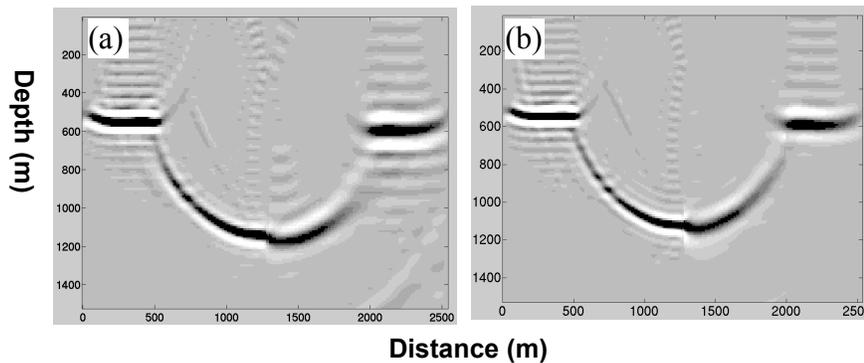


Fig. 5 Results from (a) windowed method and (b) eigenfunction decomposition method.

With a very similar model (except that there are no linear velocity variations), Stoffa et al. tested for the split-step Fourier method with the result in Figure 6.

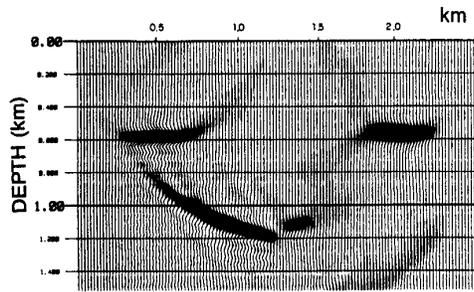


Fig. 6. Result produced by Stoffa et al. (1990).

The final test is on the data from the Marmousi model (Figure 7) for prestack migration (Bourgeois et al., 1991). The windows generated by the criteria of velocity variations are less than 10% relatively. The result is shown in Figure 8.

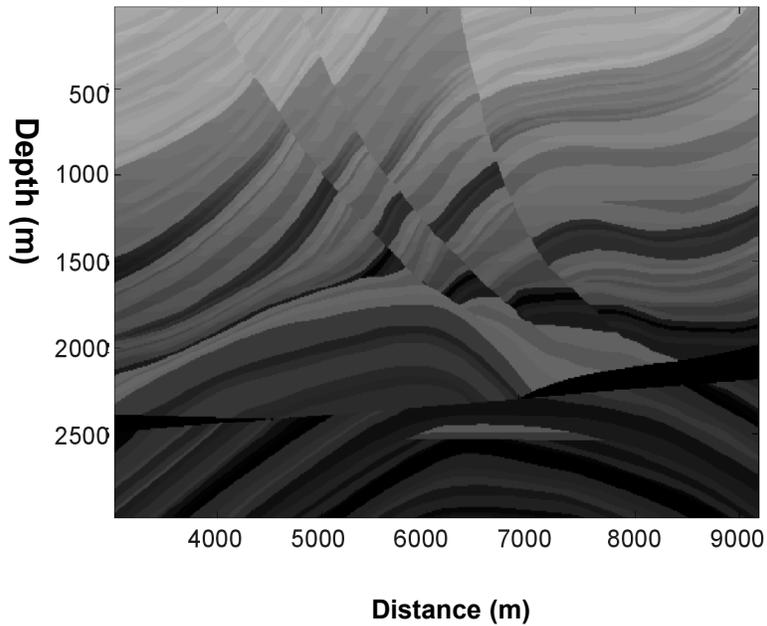


Figure 7. Marmousi model.

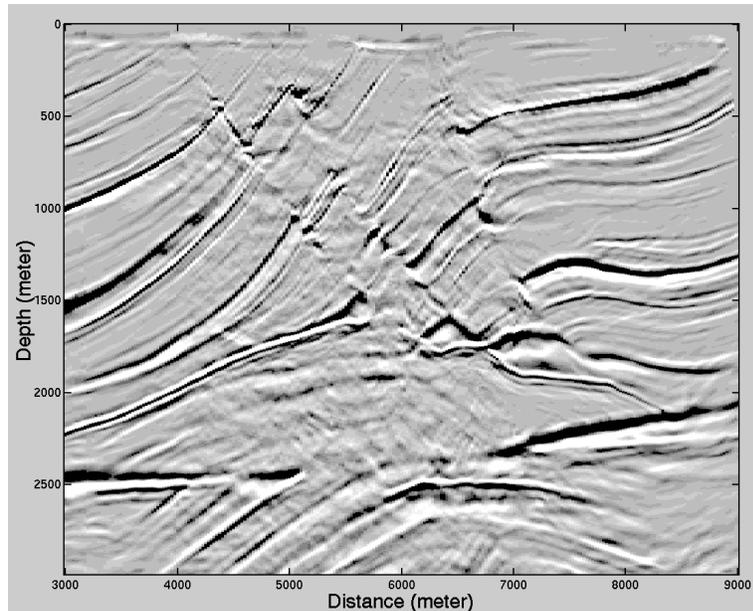


Figure 8. Migration result from Marmousi data.

Comparing the migrated result with the original velocity mode shows that all of the major seismic markers are present in the migrated image.

## CONCLUSIONS

The windowed nonstationary phase-shift method presented in this paper can handle wavefield extrapolation in strong laterally variant velocity media. It speeds up the original nonstationary phase-shift but retains accuracy. When the window is chosen as the unit function, the method comes back to the original nonstationary phase-shift method. Numerical results show that the method works well for both wavefield extrapolation and migration.

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## REFERENCES

- Bourgeois, A., Bourget, M., Lailly, P., Poulet, M., Ricarte, P., and Versteeg, R., 1990. Marmousi, model and data. In *The Marmousi experience*, EAGE, 5-9.
- Gazdag, J., 1978, Wave equation migration with the phase-shift method: *Geophysics*, **43**, 1342-1352.
- Goran E. Richards P.G, 1991. Generalized ray theory for seismic waves in structures with planar nonparallel interfaces, *B.S.S.A*, 81,
- Hardin, R.H. and Tappert, F.D., 1973. Applications of the split-step Fourier method to wave equations. *SIAM Rev.* 15, 423.
- Hildebrand, S., 1987. Local Fourier transform of the Helmholtz wave equation. *Geophysics*, 52, 1303-1305.
- Kaiser, G., 1994. *A friendly guide to wavelets*. Birkhauser Inc., Boston.

- Margrave, G.F. & Ferguson, R. J., 1999. Wavefield extrapolation by nonstationary phase shift: *Geophysics*, **64**, 1067-1078.
- Margrave, G.F. & Ferguson, R. J., 1998. Nonstationary filters, pseudodifferential operators, and their inverse: CREWES Research Report, Vol. **10**.
- Margrave, G.F. & Ferguson, R. J., 1999. Prestack depth migration by symmetric nonstationary phase shift. CREWES Research Report, Vol. **11**.
- Orsag, S. A., 1972. Comparison of pseudodifferential and spectral approximation: *Sud. Appl. Math*, 51, 253-259.
- Stoffa, P.L., Fokkema, J.T., deLuna Frire, R. M., and Kesinger, W.P., 1990. Split-step Fourier migration. *Geophysics*, 55, 410-421.
- Wapenaar, C.P.A. and Grimbergen, J.L.T., 1988. A discussion on stability analysis of wavefield depth extrapolation. Expanded abstracts 68<sup>th</sup> SEG International Convention, 1716-1719.
- Wenzel, F., 1991. Frequency-wavenumber migration in laterally heterogeneous media. *Geophysics*, 56, 1671-1673.
- Wu, R.S., 1994. Wide-angle elastic wave one-way propagation in heterogeneous media and an elastic wave complex-screen method. *J. Geophys. Res.*, 99, 751-766.
- Yao Z. and Margrave, G.F., 1999. Wavefield extrapolation: from pseudospectral method to nonstationary phase shift. CREWES Research Report, Vol. **11**
- Yao Z. and Margrave, G.F., 1999. Wavefield extrapolation in laterally inhomogeneous media by generalized eigenfunction transform method. CREWES Research Report, Vol. **11**