Errata: Reflection and transmission coefficients in T.I. media

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ABSTRACT

As the two papers that will be dealt with here, Daley and Hron (1977) [paper 1] and Graebner (1992) [paper 2], involve essentially the same topic, placement of typographical and other errors within these papers in a common location would seem appropriate. Both papers deal with particle displacement reflection and transmission coefficients at a solid/solid interface between two transversely isotropic (T.I.) media. The axes of rotational invariance in both the upper (1) and lower (2) media, separated by a plane interface, are aligned perpendicular to the plane interface.

The paper by Daley and Hron (1977) was written both to obtain reflection and transmission coefficients for this medium type for a fairly specific purpose and to show that plan-wave coefficients are associated with the zero-order approximation of an infinite asymptotic series. The 1992 paper published by Graebner, presents a much more general and possibly useful plane-wave approximation for the reflection and transmission coefficients in terms of the horizontal slowness, p, in a manner similar to those derived for isotropic media by Aki and Richards (1980). No analytic expressions for individual coefficients are given in paper 2 for the solid/solid case. Rather, the problem is set up in a manner to be solved by numerical methods using block partitioning of the original 4×4 matrix. This allows the generally complex value of a specified coefficient of the 16 possible reflection and transmission coefficients at this type of interface to be obtained over a range of slownesses.

In this report, apart from correcting the typographic errors in paper 1, the derivation of the polarization vectors in a T.I. in paper 2 will be re-examined.

PAPER 1

Since being published in 1977, the typographical errors in this paper have been located and corrected in some geophysical papers citing this work. However, in none of these have all errors been addressed. As these typographical errors do not require major derivations or explanations, they will be given in a fairly cryptic point form.

(1). The definition of $Q^{(\nu)}$ after *equation (6)* should read:

$$Q^{(\nu)} = \left[\left(A_{33}^{(\nu)} - A_{55}^{(\nu)} \right)^2 + 2A_1^{(\nu)} \sin^2 \theta_\nu + A_2^{(\nu)} \sin^4 \theta_\nu \right]. \tag{1}$$

(2). The specifications of $\cos\theta_2$ and $\cos\theta_3$, corresponding to the qP wave in medium 2 (upper medium) and the qS_V wave in medium 1, in the set of quantity definitions given between equations (14) and (15) has errors in the subscripts of $\left(\theta_j, j = 2,3\right)$ and may be correctly written as

$$\cos\theta_2 = S = \left(1 - \frac{x^2}{n^2}\right)^{1/2}$$

$$\cos\theta_3 = Q = \left(1 - k_1^2 x^2\right)^{1/2} \quad . \tag{2}$$

(3). The expressions for the eigenvalues defined in equation *(A.7)* are in error and should have the form

$$\lambda_1 = \frac{1}{2}\left\{K + \sqrt{K^2 - 4L}\right\}$$

$$\lambda_2 = \frac{1}{2}\left\{K - \sqrt{K^2 - 4L}\right\}$$

$$\lambda_3 = p_1^2 A_{66} + p_3^2 A_{55} \quad , \tag{3}$$

where

$$K = \left(A_{11} + A_{55}\right) p_1^2 + \left(A_{33} + A_{55}\right) p_3^2 \tag{4}$$

and

$$L = \left(A_{11} p_1^2 + A_{55} p_3^2\right) + \left(A_{33} p_3^2 + A_{55} p_1^2\right) - \left(A_{13} - A_{55}\right)^2 p_1^2 p_3^2 \ . \tag{5}$$

The eigenvalue λ_1 corresponds to quasi-compressional, qP, wave propagation, λ_2 to the quasi-shear, qS_V propagation mode, and λ_3 to quasi-shear, qS_H wave type.

(4). There is a sign error in the quantity defining A_1 given by equation (A.11) and should read

$$A_1 = 2\left(A_{13} + A_{55}\right)^2 - \left(A_{33} - A_{55}\right)\left(A_{11} + A_{33} - 2A_{55}\right). \tag{6}$$

These misprints were found and corrected before publication and the derivations for incidence of qP and qS_V wave types from the lower medium (medium 2) derived. The software package that resulted was compared with tables computed with manually operated devices as well as isotropic homogeneous reflection and transmission software packages and numerical methods. All inconsistencies were reconciled and the 1976 version of this code ran successfully until being replaced in the late 1980s with one more compatible with the compilers available and the dynamic range of floating-point numbers on Unix based workstations and in PC environments.

PAPER 2

The major problem in this paper, which offsets all the positive work done, is the improper derivation of the polarization vector components found in the sections entitled " ***QP - QSV* eigenvalues**" and " ***QP - QSV* eigenvectors**". What is required here is the solution of an eigenvalue problem with the resultant normalized eigenvector components corresponding to the polarization vector components. In what follows, the polarization vectors will be derived in a similar manner to that in paper 1 while the notation used in paper 2 will be retained as far as it is possible. This will allow the equations for the solid/solid reflection and transmission coefficients given in *equation (22)* in paper 2 to be correctly evaluated. The liquid/solid case is not considered here but the results of the following section are applicable to that problem also.

POLARIZATION VECTORS IN A TRANSVERSELY ISOTROPIC MEDIUM

The eigenvalue/eigenvector problem for the qP and qS_V polarization vectors, (ℓ_α, m_α) and (m_β, ℓ_β), correspond to the vectors obtained by solving the following problem for the two possible values of the eigenvalues, λ_α and λ_β. These two analytic quantities are then returned to the problem to obtain the qP and qS_V polarization vectors. (Gantmacher, 1959). This eigenvalue problem is obtained by assuming a plane-wave solution for the equations of particle motion in a T.I. medium (Cerveny and Psencik, 1972) and has the following form

$$\begin{bmatrix} A'p^2 + L'q^2 - \lambda & (F'+L')pq \\ (F'+L')pq & L'p^2 + C'q^2 - \lambda \end{bmatrix} \begin{bmatrix} \ell \\ m \end{bmatrix} = 0 \quad . \tag{7}$$

The above system has a solution, apart from the degenerate one when $(\ell, m)^T \equiv 0$, if and only if the determinant of the matrix is equal to zero. There are two solution values of λ corresponding to the qP and qS_V modes of wave propagation. In the above equation, the variable modification $A' = A/\rho$, ρ being density, has been made with A' having the dimensions of velocity squared. Similar definitions are applied to obtain C', L' and F'.[(1)]

The eigenvalues corresponding to the qP and qS_V modes of propagation are given as

$$\lambda_{qP/qS_V} = \frac{B \pm Q}{2} \quad , \tag{8}$$

with

$$B = (A+L)p^2 + (C+L)q^2 \tag{9}$$

and

[(1)] In terms of C_{ij} : $A = C_{11}$, $C = C_{33}$, $L = C_{55}$, and $F = C_{13}$. The primed quantities follow in terms of A_{ij}.

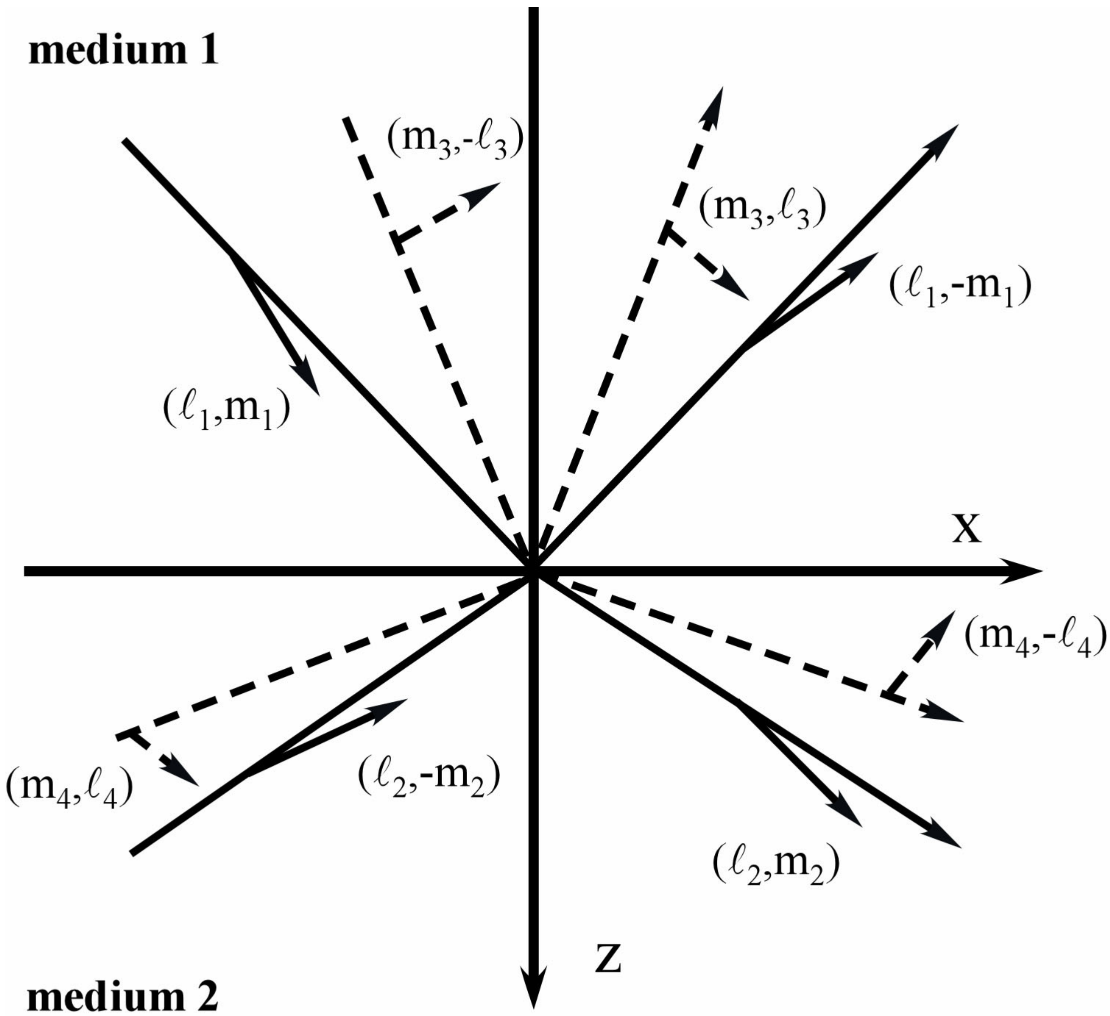

FIG. 1: Definition of the polarization vectors used in Graebner's 1992 paper. These definitions, especially the orientation of qS_V polarization vectors, must be consistent in the derivation of the particle displacement reflection and transmission coefficients and this consistent orientation must be retained if these coefficients are used together with some geometrical optics amplitude computation procedure. This holds for isotropic as well as anisotropic media. The incident wavefront normal vectors are on the left side of the vertical, *z*-axis. These include incident qP and qS_V wavefronts from both the upper and lower media. As the axes of anisotropy are parallel to the interface in both media, the magnitude of the polarization vector components in each medium are the same for the same type of wave propagation. There can be differences in the signs of the vertical components of slowness for these. However, in the geometry shown, the horizontal slownesses will all have the same sign for any given incident angle from any medium and for any wave type.

$$Q = \left[\left[\left(A - L \right) p^2 + \left(C - L \right) q^2 \right]^2 + 4 D_e p^2 q^2 \right]^{1/2} \tag{10}$$

$$D_e = \left(F + L \right)^2 - \left(A - L \right)\left(C - L \right). \tag{11}$$

The last quantity defined above is a measure of the deviation of the slowness surfaces from the ellipsoidal. $D_e \equiv 0$ specifies the degenerate ellipsoidal case.

The solution of the eigenvalue problem, results in the following expressions for the normalized components of the polarization vectors, $\boldsymbol{\ell}_{\mathbf{qP}} = \left(\ell, m \right)$ and $\boldsymbol{\ell}_{\mathbf{qS_V}} = \left(m, \ell \right)$,

$$\ell = \left[\frac{Q + \left[\left(A - L \right) p^2 - \left(C - L \right) q^2 \right]}{2Q} \right]^{1/2}, \tag{12}$$

$$m = \left[\frac{Q - \left[\left(A - L \right) p^2 - \left(C - L \right) q^2 \right]}{2Q} \right]^{1/2}. \tag{13}$$

The above expressions are accurate up to a $\pm$ sign. Each polarization vector is defined by the orientation of the vertical, z, axis, together with the user specified orientation of the zero-order (plane-wave) qS_V polarization vector and the values of the anisotropic parameters within each medium. For the situation presented here and that in paper 2, the individual definitions are shown in Figure 1 where further clarification of the notation used may be found in the caption. The orientations of the polarization vectors are consistent with those used in Aki and Richards (1980). An effort has been made in the figure to indicate that the polarization vectors are not, in general, parallel or perpendicular to the wavefront normal vector. It should be further noted that neither are they parallel or perpendicular to the ray along which energy is transported.

Specific definitions of all possible polarization vectors follow. It should be mentioned that four of the following are superscripted both as "reflected" and "transmitted". This is due to their definition depending on which is the medium of incidence. The following changes in notation will be made to simplify the equations, $qP \rightarrow \alpha$ and $qS_V \rightarrow \beta$.

$$\boldsymbol{\ell}^{\mathbf{inc}}_{\boldsymbol{\alpha}_1} = \left(\ell_{\alpha_1}, m_{\alpha_1} \right) \tag{14}$$

$$\boldsymbol{\ell}^{\mathbf{ref}}_{\boldsymbol{\alpha}_1} = \boldsymbol{\ell}^{\mathbf{trans}}_{\boldsymbol{\alpha}_1} = \left(\ell_{\alpha_1}, -m_{\alpha_1} \right) \tag{15}$$

$$\boldsymbol{\ell}^{\mathbf{inc}}_{\boldsymbol{\beta}_1} = \left(m_{\beta_1}, -\ell_{\beta_1} \right) \tag{16}$$

$$\boldsymbol{\ell}^{\mathbf{ref}}_{\boldsymbol{\beta}_1} = \boldsymbol{\ell}^{\mathbf{trans}}_{\boldsymbol{\beta}_1} = \left(m_{\beta_1}, \ell_{\beta_1} \right) \tag{17}$$

$$\boldsymbol{\ell}^{\mathbf{inc}}_{\boldsymbol{\alpha}_2} = \left(\ell_{\alpha_2}, -m_{\alpha_2} \right) \tag{18}$$

$$\boldsymbol{\ell}_{\boldsymbol{\alpha}_2}^{\mathbf{ref}} = \boldsymbol{\ell}_{\boldsymbol{\alpha}_2}^{\mathbf{trans}} = \left(\ell_{\alpha_2}, m_{\alpha_2}\right) \tag{19}$$

$$\boldsymbol{\ell}_{\boldsymbol{\beta}_2}^{\mathbf{inc}} = \left(m_{\beta_2}, \ell_{\beta_2}\right) \tag{20}$$

$$\boldsymbol{\ell}_{\boldsymbol{\beta}_2}^{\mathbf{ref}} = \boldsymbol{\ell}_{\boldsymbol{\beta}_2}^{\mathbf{trans}} = \left(m_{\beta_2}, -\ell_{\beta_2}\right). \tag{21}$$

As a simple example, consider the case of a medium that displays ellipsoidal anisotropy. In this case, D_e, the deviation of the slowness surface from the ellipsoidal is equal to zero which results in $Q = \left(A - L\right)p^2 + \left(C - L\right)q^2$. Utilizing this simplifying relation, equations (12) and (13) may be written as

$$\ell = \left[\frac{Q + \left[\left(A-L\right)p^2 - \left(C-L\right)q^2\right]}{2Q}\right]^{1/2} = \left[\frac{\left(A-L\right)p^2}{Q}\right]^{1/2} \tag{22}$$

$$m = \left[\frac{Q - \left[\left(A-L\right)p^2 - \left(C-L\right)q^2\right]}{2Q}\right]^{1/2} = \left[\frac{\left(C-L\right)q^2}{Q}\right]^{1/2}. \tag{23}$$

Given the definition of the vertical and horizontal components of slowness in terms of the phase (wavefront normal) angle θ as

$$p = \frac{\sin\theta}{V_N\left(\theta\right)} \tag{24}$$

and

$$q = \frac{\cos\theta}{V_N\left(\theta\right)} \tag{25}$$

where $V_N\left(\theta\right)$ is the phase velocity which for the qP ellipsoidal case is

$$V_N^{QP}\left(\theta_{QP}\right) = \left(A'\sin^2\theta_{QP} + C'\cos^2\theta_{QP}\right)^{1/2} \tag{26}$$

and for the qS_V case is

$$V_N^{QSV}\left(\theta_{QSV}\right) = \sqrt{L'}. \tag{27}$$

The resulting polarization vectors are then

$$\ell_{\alpha} = \left[\left(\frac{A' - L'}{A' \sin^2 \theta_{qP} + C' \sin^2 \theta_{qP} - L'} \right)^{1/2} \sin \theta_{qP}, \left(\frac{C' - L'}{A' \sin^2 \theta_{qP} + C' \sin^2 \theta_{qP} - L'} \right)^{1/2} \cos \theta_{qP} \right] \quad (28)$$

and

$$\ell_{\beta} = \left[\left(\frac{C' - L'}{A' \sin^2 \theta_{qSv} + C' \sin^2 \theta_{qSv} - L'} \right)^{1/2} \cos \theta_{qSv}, \left(\frac{A' - L'}{A' \sin^2 \theta_{qSv} + C' \sin^2 \theta_{qSv} - L'} \right)^{1/2} \sin \theta_{qSv} \right] \quad (29)$$

The directional definitions of ℓ_{α} and ℓ_{β} may be inferred from equations (14) – (21) and Figure 1.

DISCUSSION AND CONCLUSIONS

Once the modifications presented here are made, *equation (22)* and related equations in paper 2 may be used for the computation of reflection and transmission coefficients at a solid/solid interface between two transversely isotropic media. Analytic expressions for the 16 individual coefficients have been obtained and a subroutine set written. This most current code has been compared with previously derived formulae for these coefficients, both analytic and numerical, with favourable results.

A few examples using this recently developed code will be given for reference purposes. The model used for the upper (1) layer is one devised for a fairly specific purpose and used in the paper of Mikhailenko (1985). The anisotropic parameters were chosen such that the qS_V energy (ray) surface has triplications. It, together with the qP ray surface, is shown in Figure 1. The lower (2) medium is a relatively weak anisotropic layer with both the qP and the qS_V slowness surfaces being convex indicating that no triplications occur on the ray surfaces. The model parameters are given in Table 1.

The P_1P_1 reflection coefficient with amplitude and phase plotted against phase angle of incidence is shown in Figure 1. Incidence and reflection are both in layer 1.

The second example shown is the P_1S_1 reflection coefficient with incidence from layer 1. Only the amplitudes are displayed and are done so against both the phase and ray incident angles. The complexity of the coefficient due to the triplications on the qS_V ray surface is clearly evident.

Table 1. The anisotropic parameters of the two transversely isotropic halfspaces used in the computation of the figures that follow. The dimensions of the A_{ij} are km^2/s^2 and density has the dimension gm/cm^3.

Layer	A_{11}	A_{33}	A_{44}	A_{13}	ρ
1	14.8246	12.2080	1.3059	6.7032	2.0
2	30.0000	25.0000	7.4000	12.2750	2.2

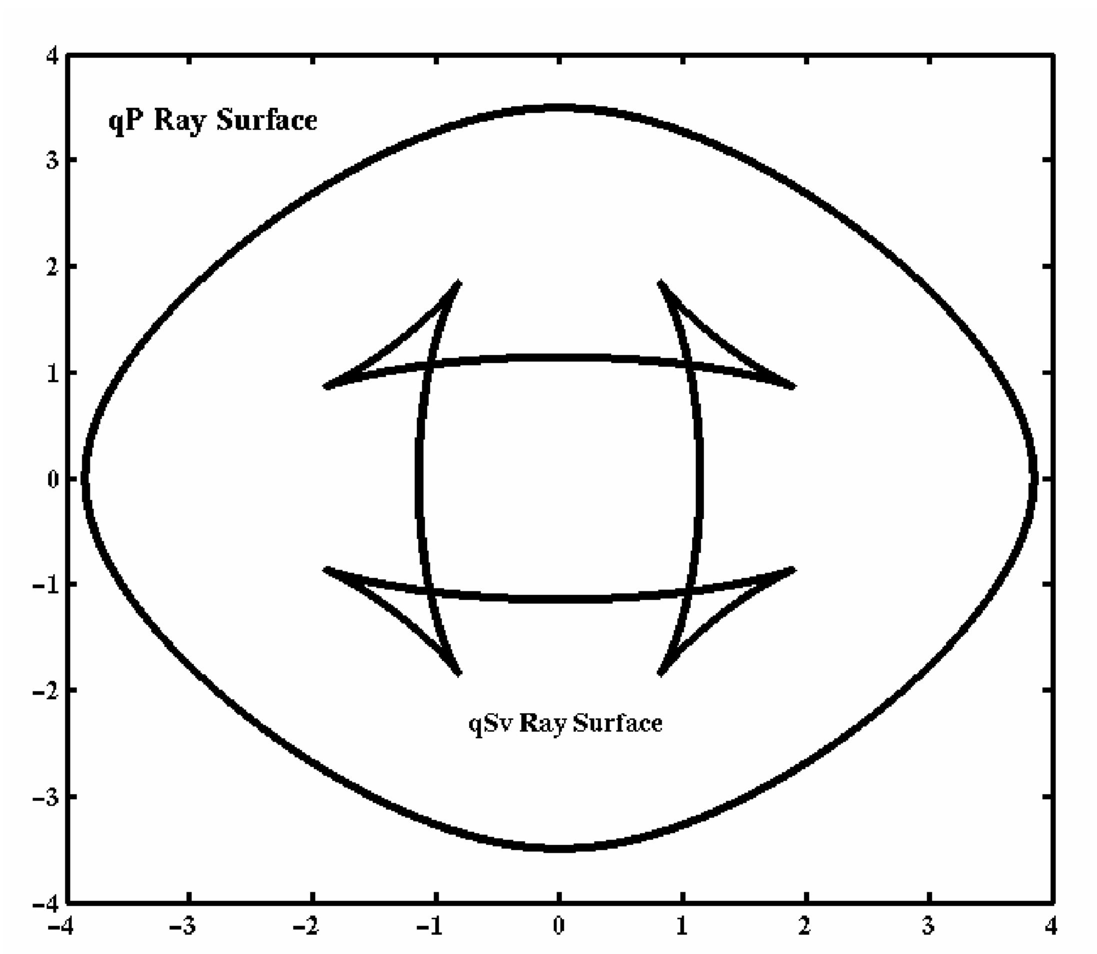

FIG. 2: The *qP* and *qSv* ray surfaces for layer 1 of the model described in Table 1.

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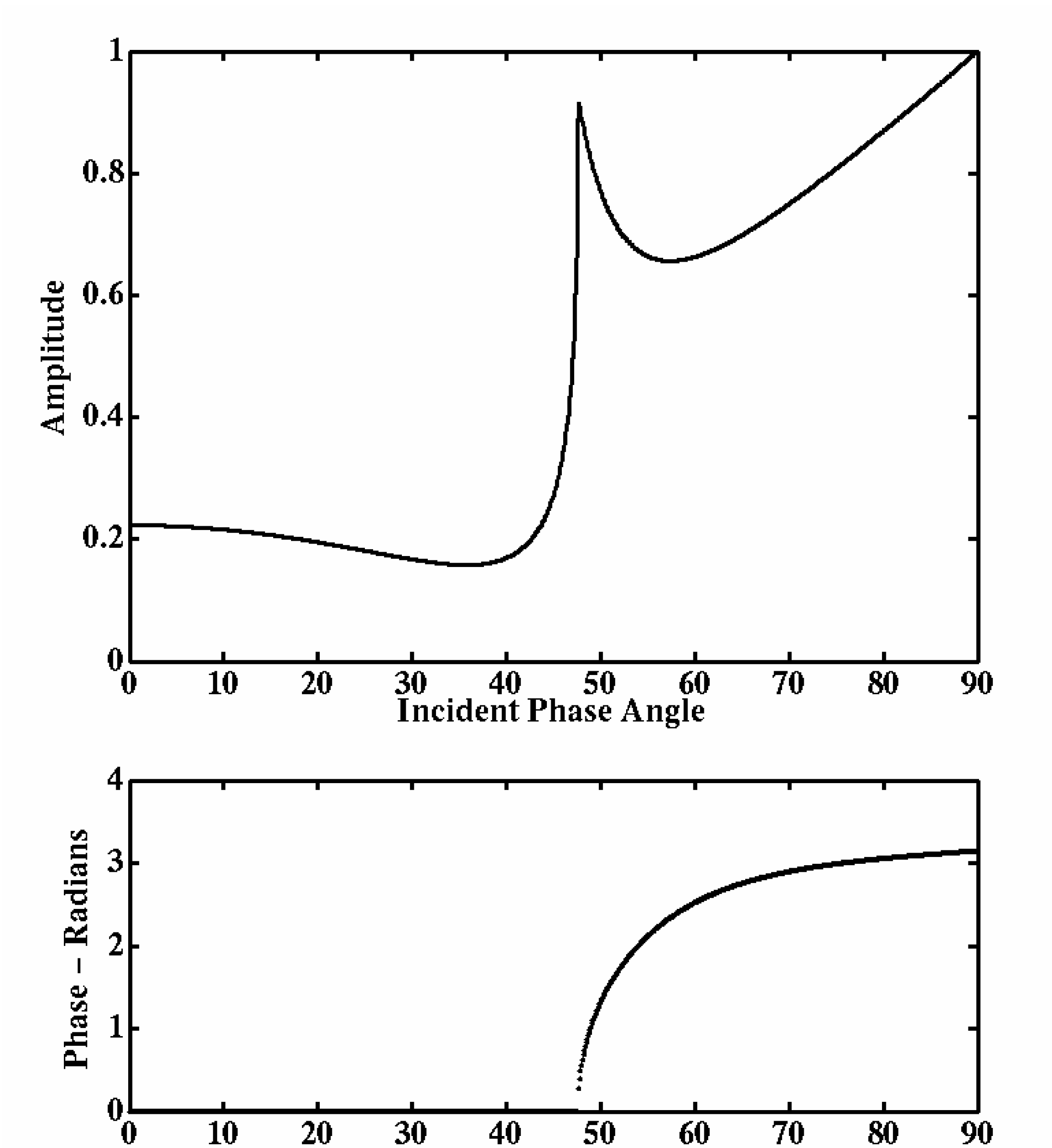

FIG. 3: The P_1P_1 reflection coefficient, amplitude, and phase versus phase angle of incidence for the model given in Table 1.

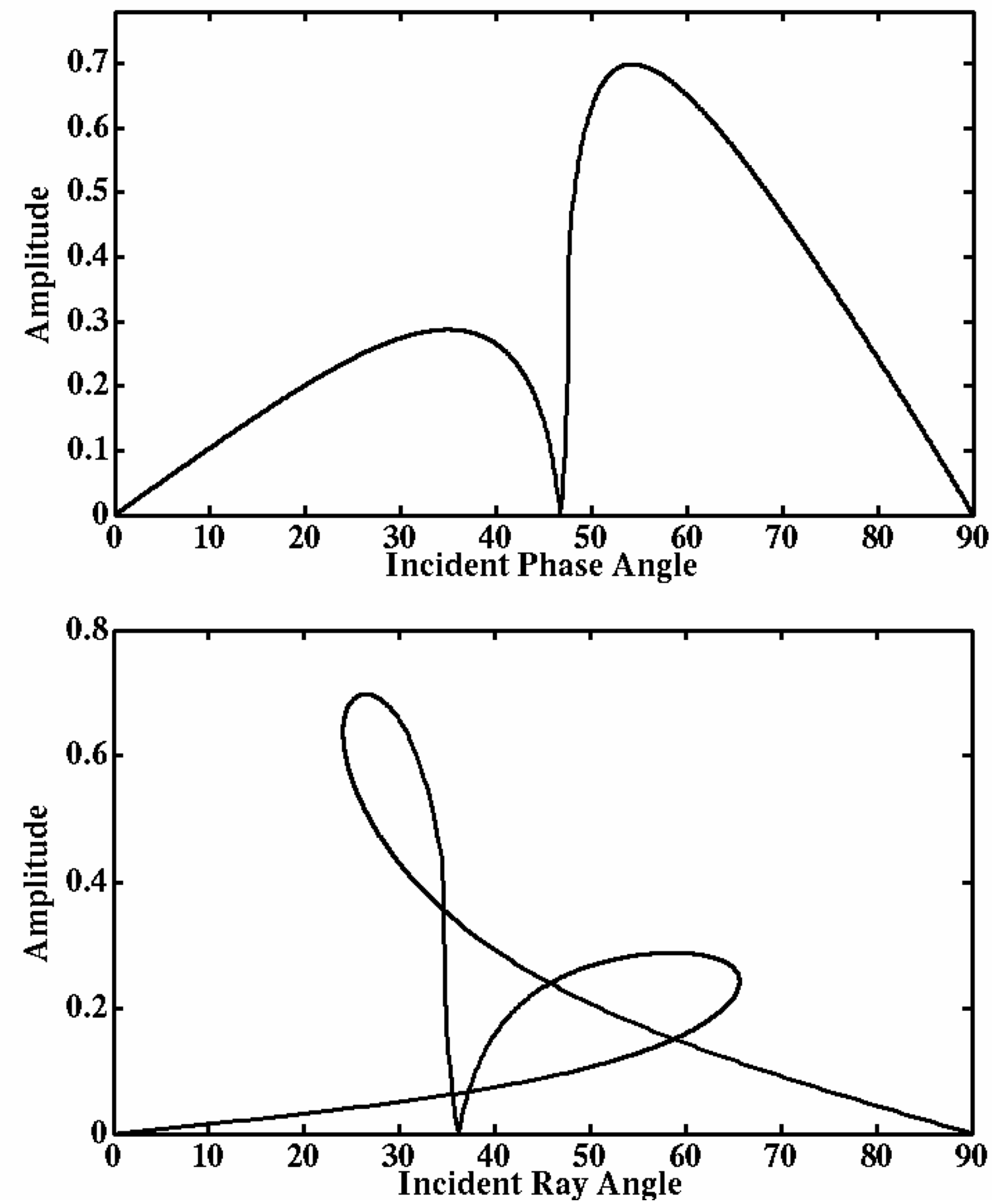

FIG. 4: Comparison of the amplitudes of the P_1S_1 reflection coefficient for the model given in Table 1 plotted against both phase angle of incidence and ray angle of incidence.