2D wave-equation migration by joint finite element method and finite difference method

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ABSTRACT

A new method of migration using the finite element method (FEM) and the finite difference method (FDM) is jointly used in the spatial domain. It has been applied to solve a time relay 2D wave equation. By using the semi-discretization technique of FEM in the spatial domain, the origin problem can be written as a coupled system of lower dimensions partial differential equations (PDEs) that continuously depend upon time and space. FDM is used to solve these PDEs. The concept and theory of this method are also discussed in this paper. Two numerical examples of 2D wave-equation migration show the successful result and its potential application.

INTRODUCTION

The finite element–finite difference method (FE–FDM) is one of the numerical methods using FEM and FDM in the spatial domain to solve partial differential equations. FE-FDM uses FEM in some dimensions and FDM in the remaining dimensions and in the time domain. The FE-FDM has strong resemblance to a number of numerical methods such as the finite difference method and the finite element method. A brief emphasis on the basic differences between FE-FDM and the above mentioned methods is as follows:

FEM fully discretizes a static problem into a system of algebraic equations with discrete nodal values as the basic unknowns. For the time relay problem, FEM fully discretizes it in spatial domain into ODEs and solves them with the FD method (Hughes, 1987), whereas the FE-FDM semi-discretizes the PDE using FEM in the spatial domain into a coupled system of PDEs. These PDEs still continuously depend upon both time and space (although not all the space dimension), and are solved with FD method. Thus, the strengths of FEM, the adaptation to arbitrary domain, boundary, material and loading are retained. The shortcomings of FEM, such as large demand on computer memory and high computation costs are reduced because of the semi-discretization. Compared with the FD method, the computation precision is increased by FEM semi-discretization. The technique of FD for solving PDEs in lower dimensions can decrease frequency dispersion in space and has looser conditions of stability for explicit FD schemes.

In this paper, the basic concept and theory of the finite element–finite difference method are described through the 2D wave equation. Two numerical examples of 2D wave equation migration are given as well to demonstrate the tremendous performance of this method.

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PRINCIPLE

Consider the hyperbolic model problem, with the 2D scalar wave equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial z^2}=\frac{1}{a^2(x,z)}\frac{\partial^2 u}{\partial t^2}\text{, in}\,\Omega \tag{1a}$$

Here $u(x,z,t)$ denotes the wave disturbance at horizontal (lateral) coordinate x, vertical (depth) coordinate z (where the z axis points downward) and time t, respectively. a(x, z) is the medium velocity. We assume a boundary condition (B. C.) of the form:

$$u(x,z,t)=\begin{cases}\varphi(x,t) & z=0\\ 0 & z\neq 0\end{cases}\text{, in }\partial\Omega \tag{1b}$$

and the initial condition:

$$u(x,z,t=T)=\phi(x,z)\,,\dot{u}(x,z,t=T)=0\text{ , in}\,\Omega\,. \tag{1c}$$

The two-dimensional domain Ω is bounded by the piecewise smooth boundary ($\partial\Omega$). The purpose of wave-equation migration is to solve the above equation so that the recorded wave field at t=T can propagate back to t=0; hence the reflected wave lies at the reflection interface (Yilmaz, 1987). FE-FDM discretizes (1a) in the x-coordinate using FEM, and solves the remaining equations in the z-t coordinates using the FD method.

2.1 FEM semi-discretization in x-coordinate

P1 denotes the partial differential equation (1). P2 denotes the corresponding Galerkin method of P1. P2 is:

Find $u\in S_\varphi^1$, such that for all $v\in S_0^1$

$$D(u,v)-F(v)=0\,. \tag{2}$$

Here

$$S_{\varphi}^{1} = \left\{ u \mid \int \left[u^2 + \left(\frac{\partial u}{\partial x} \right)^2 \right] dx < \infty,\ u(x,z,t) = \begin{cases} \varphi(x,t) & z = 0 \\ 0 & z \neq 0 \end{cases}, \text{in } \partial\Omega \right\},$$

$$S_{0}^{1} = \left\{ v \mid \int \left[v^2 + \left(\frac{\partial v}{\partial x} \right)^2 \right] dx < \infty,\ v(\partial\Omega) = 0 \right\},$$

$$D(u,v) = \iint \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2} - \frac{1}{a^2(x,z)} \frac{\partial^2 u}{\partial t^2} \right] v dx,\quad F(v) = 0$$

$D(u,v)$ can be rewritten as

$$D(u,v) = \int_{\Omega} \left[\frac{\partial u}{\partial x} \frac{\partial v}{\partial x} - \frac{\partial^2 u}{\partial z^2} v + \frac{1}{a^2(x,z)} \frac{\partial^2 u}{\partial t^2} v \right] dx\,. \tag{3}$$

Semi-discretizing the horizontal coordinate (x) in the region of [0, X], one constructs finite element function space as

$$u_h(x,z,t) = \sum_{i=1}^{N} u_i(t,z) N_i(x)\,, \tag{4a}$$

$$\frac{\partial}{\partial x} u_h(x,z,t) = \sum_{i=1}^{N} u_i(t,z) \frac{d}{dx} N_i(x) = \sum_{i=1}^{N} u_i(t,z) B_i(x)\,, \tag{4b}$$

where N is the nodal numbers. By substituting equation (3) and (4) into (2), one gets the discrete style description of P2.

$$D(u_h, v_h) = \sum_{n=1}^{NE} \int_{\Omega} \left[v_e^T B^T B u_e - v_e^T N^T N \frac{\partial^2 u_e}{\partial z^2} + \frac{1}{a^2(x,z)} v_e^T N^T N \frac{\partial^2 u_e}{\partial t^2} \right] dx = 0\,,$$

NE is the total numbers of all the nodes, u_e, v_e is the each cell vector, e means each cell and this expression can be simplified. For the reason of function v is arbitrary, one obtains semi-discretized PDEs as

$$M \frac{\partial^2 u}{\partial t^2} + K u = H \frac{\partial^2 u}{\partial z^2}\,, \tag{5a}$$

with boundary condition (B. C.)

$$u(z=0,t)=g(t)\,, \tag{5b}$$

and initial conditions

$$u(z,t=T)=f(z),\ \dot{u}(z,t=T)=0\,, \tag{5c}$$

where $g(t)$ and $f(z)$ are the discretization of $\varphi(x,t)$ and $\phi(x,z)$ respectively.

$$M=\sum_{n=1}^{NE}M_e\,,\quad K=\sum_{n=1}^{NE}K_e\,,\quad H=\sum_{n=1}^{NE}H_e\,, \tag{5d}$$

$$M_e=\int_e \frac{1}{a^2(x,z)}N^T N dx\,, K_e=\int_e B^T B dx\,,\ H_e=\int_e N^T N dx\,, \tag{5e}$$

It can be seen that the matrices M, K and H are all symmetric. M and H are positive-definite, and K is positive-semidefinite. The PDEs are model hyperbolic equations when the velocity is constant because the matrices M and H can be diagonalized at the same time under this condition. It should be emphasized that only the matrix M varies with depth.

2.2 FDM solution of matrix PDEs

One of the explicit schemes, the five-point central scheme, is selected to solve this problem. The difference equation has the form

$$Mu[i]_j^{n-1}=Au[i]_j^n+B(u[i]_{j+1}^n+u[i]_{j-1}^n)-Mu[i]_j^{n+1}\,. \tag{6}$$

Here

$$A=2M-\tau^2K-\frac{2\tau^2}{h^2}H\,,\ B=\frac{2\tau^2}{h^2}H\,,$$

where τ, and h are the time and space steps, assumed constant, and i, j, k are the discrete denotation of lateral direction, depth direction and time, respectively. The local truncation error of this scheme has the form of O(τ^2+h^2) (Durran, 1999).

2.3 Stability discussion of matrix PDEs

The stability of wave equation with BC and IC is much complicated. For this problem, the stability is hard because it is related to the FEM semi-discrete scheme and the form of the interpolation function.

The scheme stability analysis of the simplest condition is discussed here. Consider piecewise linear interpolation function. This problem has only one element. The element length is l, the velocity is a, x_i is the each node coordinate value. The interpolation function (assume n=2, x_1, x_2 is the two node coordinate value)

$$N(x) = (\frac{1}{2}(1+\xi),\ \frac{1}{2}(1-\xi))\text{, here } \xi = 2\frac{x_i - x_c}{l},\ x_c = \frac{x_1 + x_2}{2},$$

The coefficient matrices of equation (5a) is

$$M = \frac{l}{3a^2}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, K = \frac{1}{2l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, H = \frac{l}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

By diagonalizing these equations, one gets two individual equations

$$\frac{\partial^2 u_1}{\partial t^2} + \frac{3}{l^2}u_1 = a^2\frac{\partial^2 u_1}{\partial z^2}, \tag{7a}$$

$$\frac{\partial^2 u_2}{\partial t^2} = a^2\frac{\partial^2 u_2}{\partial z^2} \tag{7b}$$

By using the central scheme in both the time and space, the stability condition of equation (7b) is $a^2\lambda^2 < 1$. Here τ and h are the time and space steps, and $\lambda = \tau / h$. Considering equation (7a) only, we use the Fourier analysis method. The amplification matrix has the form

$$G = \begin{bmatrix} 2 - 4a^2\lambda^2 \sin^2\frac{kh}{2} - 3\frac{\tau^2}{l^2} & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} b & c \\ 1 & 0 \end{bmatrix},$$

Because of $|c| = 1$ and $|b| \le 1 - c$, the eigenvalue of the amplification matrix $|\mu(G)| \le 1$ meets the sufficient condition of scheme stability. From $|b| \le 1 - c$, it can be obtained that

$$a^2\lambda^2 + \frac{3\tau^2}{4l^2} \le 1. \tag{8}$$

It can be verified that the equation (8) is the sufficient and necessary stability condition for equation (7a). Especially, when the element length is equal to the depth step ($l = h$), the stability condition becomes $(a^2 + \frac{3}{4})\lambda^2 < 1$. When the velocity of the wave equation is much larger than 3/4, as in real rock (where it is about several hundred metres per second), the stability condition of $a^2\lambda^2 < 1$ for equation (7b) is still valid for (7a). This condition is much looser than that ($a^2\lambda^2 < 1/2$) of the 2D space and time central scheme FD method (Lu and Guan, 1987). It is to mean that the time-step restriction imposed in FE-FDM is often much smaller than that needed for accuracy in FDM.

In the numerical test, to improve the accuracy of the computation, a one dimensional element is chosen and the node is selected as 3(i.e. n=3). The interpolation function (assume n=3, x_1, x_2, x_3 is the three node coordinate value)

$$N(x) = (\frac{1}{2}\xi(1-\xi), 1-\xi^2, \frac{1}{2}\xi(\xi+1)),$$

where $\xi = 2\frac{x_i - x_c}{l}$, $x_c = \frac{x_1 + x_3}{2}$.

NUMERICAL EXAMPLES

In order to test the validity of the FE-FDM, two numerical examples are chosen. One is the typical impulse response migration, and the other is steep obliquity migration. To show the advantage and potential of FE-FDM, it is compared with the FKFD migration method.

3.1 Impulse response migration

3.1.1 Model introduction

Impulse response, the fundamental model of migration (Robinson, 1983) is selected to test the FE-FDM migration algorithm. The poststack profile in a constant-velocity medium (a=2000m/s) is shown as Figure 1. An impulse is located at x=1000m (lateral) and t=0.8s (time). This means that the reflection interface is a semicircle with a radius of 800m and center at x=1000 on the ground. The migration steps of time, lateral and depth are 4ms, 10m and 10m.

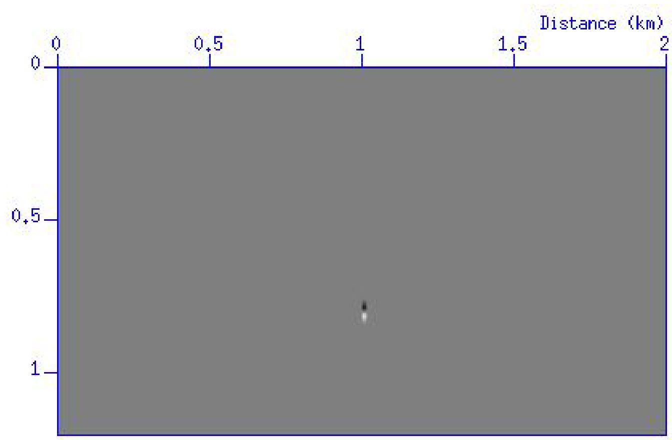

FIG. 1. Impulse record. (The impulse is located at 1000m, 0.8s)

The ideal result is shown in Figure 2a, which we obtained by placing the impulse to the reflection interface uniformly. The result of the FE-FDM method and the F-K domain FD method (90 degree) is shown as Figures 2a and 2b. The program used to generate the results of Fig.2a and 2b is part of the free software package Seismic UNIX (SU) produced by the Center for Wave Phenomena (CWP), Colorado School of Mines.

3.1.2 Remark

It can be seen from Figure 2 that all the two methods can image the semicircle reflection interface correctly. The steeply dipping part of the interface (near ground) is well imaged by FE-FDM migration (Figure 2b), but not well by omig-x domain FD migration. The result of FD migration (Figure 2b) has a lot of noise because of the spatial dispersion.

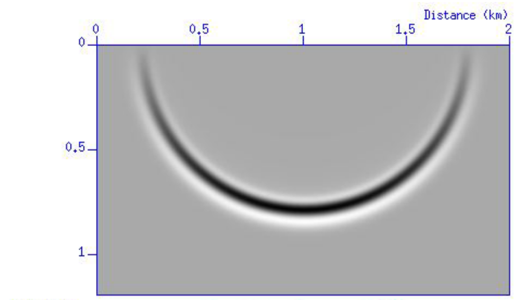

FIG. 2a. Result of FE-FDM

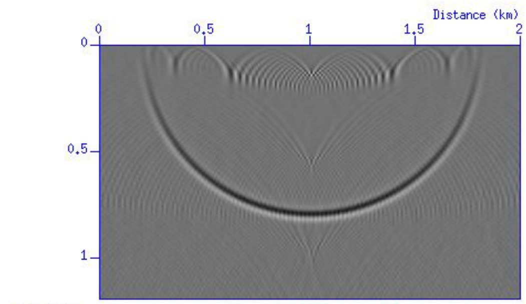

FIG. 2b. Result of FKFDM

Selecting the centre traces (x=1000) of Figure 2, and translating into temporal domain and drawing together, we obtain Figure 3. The impulse shapes of the two migration results both differ from the original one. The obvious drawback of the result of FE-FDM migration is the shape of impulse becoming low and wide because of the numerical dissipation. The dissipation of FE-FDM migration leads to a decrease of spatial resolution. Changing the scheme is one possible method to improve accuracy. Massive pseudo-waves with high amplitude, such as at t=0.1 and t=0.25, appear in the result of the omig-x domain FD method because of the numerical dispersion. The dispersion in omig-x domain FD migration could lead to wrong interpretation results.

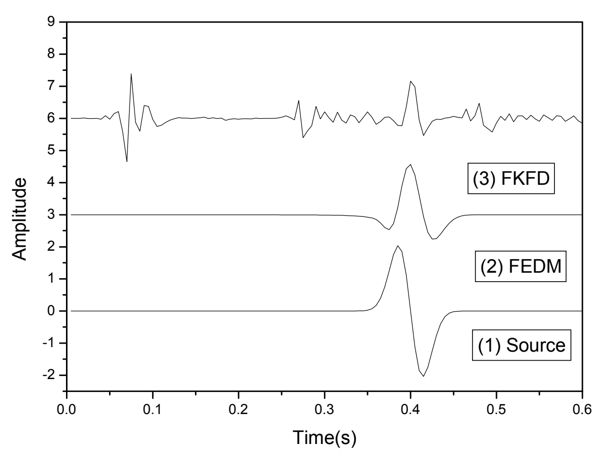

FIG. 3. Impulse records at 1000m in Figure 2(a, b) compared to the original impulse from Figure 1

3.1.3 Efficiency comparison

Adopting the impulse response model, we compared the expenditure of FE-FDM and omig-x domain FD migration on a PC (Table 1).

Table 1: The efficiency comparison between the two methods

Method	Time(s)	Memory(Mb)
FEFD	26	2.176
FKFD	31	1.28

From the results shown in Table 1, one can see the FE-FDM migration is the faster than the FK-FD method, but also occupies nearly twice the memory.

3.2 Steep Oblique Migration

3.2.1 Model introduction

The model for this section is shown in Figure 4. The velocity of the model increases both laterally and with depth direction. The velocity at the top left corner is 3600m/s, and the at the bottom right it is 4600m/s. There are two reflection interfaces, one with a decline of 45 degrees, and the other being a flat interface.

The seismograph is computed by the FDM module of the SU Software Kit, and is displayed in Figure 5. From it, one can see that the event of the flat interface is overlain by a declining interface and an inhomogenous medium. To remove these influences, the FE-FDM and the FK-FDM are used to test the effect of migration. Figure 6 is the result of FEFDM, and Figure 7 is the result of FK-FDM.

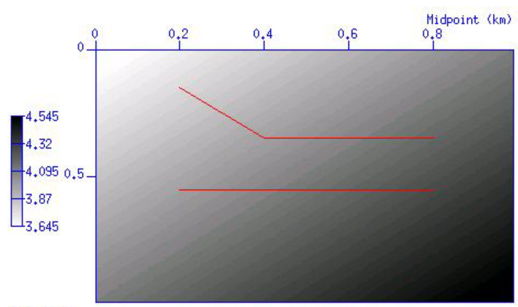

FIG. 4. Steep oblique Model

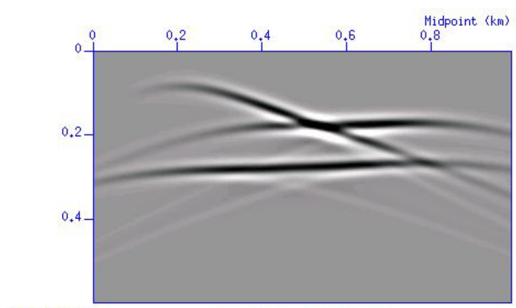

FIG. 5. The seismograph in use of FDM

3.2.2 Remark

It can be seen from Figure 5 and Figure 6 that both methods can effectively image the position of layers and have good correspondence with the model. In addition, the two methods can accurately image the geometry under the oblique reflector and the inhomogeneity.

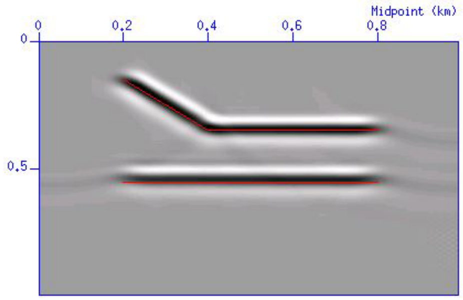

FIG. 6. The result of FE-FDM

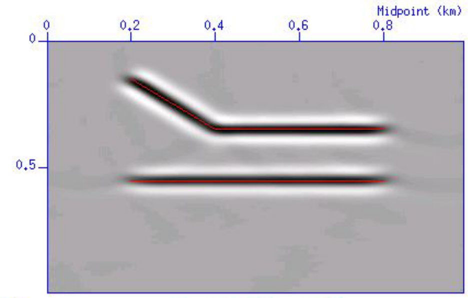

FIG. 7. The result of FK-FDM

3.3 Discussion

Omiga-x domain FD migration is a method for the one-way approximation wave equation. The equation used in this paper is accurate for propagation directions to 90 degrees. FT and IFT are applied for time only. The FD method is used in omig-x domain for wavefield extrapolation (Lee and Suh, 1985). This algorithm fits lateral velocity variation and complex interfaces so well that it has become the most popular migration method today. The drawbacks of it are the high computation costs, spatial dispersion, and its limited ability to image steeply dipping interface.

FEM migration is a highly accurate method for the problem of arbitrary shaped domains, lateral velocity variations, and complex and dipping interfaces (Teng and Dai, 1989). But it is not widely used in seismic exploration because of its large demands on computational costs and computer memory. FE-FDM migration inherits all the advantages of FEM migration presented above. The computational efficiency is improved through spatial domain semi-discretization. As shown above, the FE-FDM migration can successfully be applied to field data.

CONCLUSIONS

A numerical method named finite element–finite difference method (FE-FDM) for the solution of time relay partial differential equations such as parabolic and hyperbolic model equations is presented in this paper.

As the numerical examples, 2D scalar wave equation reverse-time depth migration has been shown above, and it is encouraging that the result is accurate and effective enough for steeply dipping interface imaging.

This method combines FEM and FDM based upon the semi-discretization of the spatial domain. The main strengths of FEM (adaptation to arbitrary domain and accuracy) and FEM (computation efficiency) are inherited. FE-FDM can be used to get accurate results for migration more accurately than the FD method and much more quickly than FEM. At the same time, it can be used to implement elastic-wave equation migration, or simulate wave propagation. It is therefore a useful and promising numerical method.

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