Alternative linearized expressions for $q_P$, $q_{S_1}$ and $q_{S_2}$ phase velocities in a weakly anisotropic orthorhombic medium

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ABSTRACT

Alternative linearized approximations of the phase velocities for the quasi-compressional, $q_P$, and two quasi-shear wave types, $q_{S_1}$ and $q_{S_2}$, in an orthorhombic medium are presented. Algebraic manipulation of the formulae obtained from the standard linearization technique is done so that the phase velocities are in the form consisting of the most degenerate cases of phase velocities in an orthorhombic medium (ellipsoids) plus anellipsoidal correction (perturbation) terms to compensate for the deviation from the degenerate orthorhombic case. The quantities in the formulae for the phase velocities all have physical interpretations, that is, they can all be associated with some physically realizable and measurable quantity. After obtaining these intermediate linearized expressions for the $q_P$, $q_{S_1}$ and $q_{S_2}$ phase velocities, further approximations are made to obtain the equivalent of what are termed fully linearized formulae. This includes the introduction of an isotropic background velocity, $\alpha$, for the $q_P$ and $\beta$ in the $q_{S_1}$ and $q_{S_2}$ equations. A comparison of the intermediate linearized approximations for the phase velocities with the exact formulae are presented in a series of figures.

INTRODUCTION

The specific problem of obtaining linearized approximations of the phase velocities for the quasi-compressional, $q_P$ and the two quasi-shear wave modes, $q_{S_1}$ and $q_{S_2}$, in an orthorhombic anisotropic medium is addressed. The linearization formulae for these phase velocities in a general anisotropic medium may be found in the works of Backus (1965), Every (1980), Jech and Pšenčík (1989), Mensch and Rasolofosaon, (1997), and Pšenčík, and Gajewski (1998), among others. Once linearized approximations are obtained, a rearrangement of terms is done to put the anisotropic coefficients from the original formulae for the $q_P$ and $q_{S_1}$ and $q_{S_2}$ phase velocities into alternate configurations. This is done so that each of the terms, or individual collection of terms in the expressions for the phase velocities, has a physical meaning or can be associated with some geometrical formalism. What results are approximations that are less “weak anisotropic” than they are “weak anellipsoidal” approximations.

Exact expressions for the $q_P$, $q_{S_1}$ and $q_{S_2}$ phase velocities in an orthorhombic medium are presented in Every (1980), and Schoenberg and Helbig (1997) as the solutions of a cubic equation. An approximation of the above exact equation by Tsvankin (1997) produced a linearized formula for the $q_P$ phase velocity in a weakly anisotropic medium which was similar to that obtained by Mensch and Rasolofosaon, (1997) who also treated the $q_S$ phase velocities. Sayers (1994) developed an approximate expression for the $q_P$ phase velocity using an expansion in spherical harmonics. In the work of Pšenčík, and Gajewski (1998) the case of a general weak anisotropic medium is
considered from a linearization perspective based on perturbation methods and arrived at the same results as those presented in Backus (1965). Apart from phase velocities, quantities such as polarization vectors are discussed for $qP$ waves in general, and a number of specific anisotropic media types, including orthorhombic and two transversely isotropic orientation categories are discussed in the paper of Pšenčík and Gajewski (1998).

The difference in the formulae for the $qP$, $qS_1$ and $qS_2$ phase velocities in an orthorhombic medium derived here, when compared to the results obtained from the linearization process, are that they all consist of two parts: an ellipsoidal expression and anellipsoidal correction (perturbation) terms. Each term, or specific collection of terms, within these phase velocity formulae can be associated with some physical or geometrical quantity that is measurable from, for example, travel time data. This modification required that some moderately complicated algebraic manipulations of the linearized expressions be undertaken.

It has been shown by a number of authors (for example; Song and Every, 2000, Pšenčík et al., 2000) that of the 9 anisotropic parameters defining an orthorhombic medium only 3 of these, $A_{11}$, $A_{22}$ and $A_{33}$, can be obtained from the inversion of $qP$ phase velocity measurements, together with 3 other quantities related to the deviation of the $qP$ wave surface from the ellipsoidal. To fully specify individual anisotropic parameters within this analysis, the inversion of shear wave data producing the terms $A_{44}$, $A_{55}$ and $A_{66}$ is required, together with the abovementioned anellipsoidal deviation terms from the $qP$ inversion process, to obtain the anisotropic parameters $A_{12}$, $A_{13}$ and $A_{23}$. This facilitates undertakings such as determining methods to pursue for the inversion of phase velocity data to obtain the remaining anisotropic parameters that define the medium. Once these rearranged formulations are obtained, other matters are addressed. This includes ensuring that associated individual quantities that are to be determined in the inversion process are all of the same approximate magnitude.

**LINEARIZED qP PHASE VELOCITY**

The linearized expression for the $qP$ phase velocity in a general anisotropic medium is given by the apparently simple formula presented in the work of Backus (1965), who showed that the linearization process was a direct consequence of first order perturbation theory, as

$$v_{qP}^2 (\mathbf{n}) = \alpha_{ijk} n_i n_j n_k n_\ell \quad (i,j,k,\ell = 1,2,3)$$

(1)

where summation over repeated indices is assumed. Others who have pursued a similar method of development for phase velocity determination as well as ancillary quantities, such as polarization vectors, are Červený and Jech (1982), Jech and Pšenčík (1989), and Pšenčík and Gajewski (1989). These related extensions will not be considered here.

The quantities $n_i$ in equation (1) are the components of the unit phase normal vector, which is defined as
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\[ \mathbf{n} = (n_1, n_2, n_3) = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \]  

(2)

where \( \theta \) is the polar angle measured from the positive \( x_3 \) axis \( (0 \leq \theta \leq \pi) \) and \( \phi \) is the azimuthal angle measured in a positive sense from the \( x_1 \) axis \( (0 \leq \phi \leq 2\pi) \).

The density \( (\rho) \) normalized anisotropic parameters, \( a_{ijkl} = c_{ijkl} / \rho \), which have the dimensions of velocity squared may be transformed to Voigt notation, \( A_{mn} \), using the scheme \( a_{ijkl} \rightarrow A_{mijn} \) with the following substitutions

\[ 11 \rightarrow 1 \quad 22 \rightarrow 2 \quad 33 \rightarrow 3 \quad 31 = 13 \rightarrow 5 \quad 21 = 12 \rightarrow 6 \quad 32 = 23 \rightarrow 4 \]  

(3)

The linearized expression, from equation (1), for the \( qP \) phase velocity in an orthorhombic medium, and introducing Voigt notation, is

\[ v_{qP}^2 (\mathbf{n}) = A_{11} n_1^2 + A_{22} n_2^2 + A_{33} n_3^2 + 2(A_{12} + 2A_{66}) n_1^2 n_2^2 + 2(A_{13} + 2A_{55}) n_1^2 n_3^2 + 2(A_{23} + 2A_{44}) n_2^2 n_3^2. \]  

(4)

Further treatment of the \( qP \) phase velocity will be given in some detail. The same detail will not be afforded to the quasi-shear phase velocities as the methods are similar.

The procedure for obtaining an alternate expression of the linearized \( qP \) phase velocity, as an ellipsoid with 3 anellipsoidal correction terms, is initiated by adding and subtracting the expression

\[ (A_{11} + A_{22}) n_1^2 n_2^2 + (A_{11} + A_{33}) n_1^2 n_3^2 + (A_{22} + A_{33}) n_2^2 n_3^2 \]  

(5)

from equation (4). This results in the formula

\[ v_{qP}^2 (\mathbf{n}) = A_{11} n_1^2 + A_{22} n_2^2 + A_{33} n_3^2 + \left[ (A_{11} + A_{22}) n_1^2 n_2^2 + (A_{11} + A_{33}) n_1^2 n_3^2 + (A_{22} + A_{33}) n_2^2 n_3^2 \right] - 2\left[ (A_{12} + 2A_{66}) - (A_{11} + A_{22}) \right] n_1^2 n_2^2 + 2\left[ (A_{13} + 2A_{55}) - (A_{11} + A_{33}) \right] n_1^2 n_3^2 - 2\left[ (A_{23} + 2A_{44}) - (A_{22} + A_{33}) \right] n_2^2 n_3^2. \]  

(6)

Simplifying the above equation using the properties of the vector \( \mathbf{n} \), given by equation (2), yields

\[ v_{qP}^2 (\mathbf{n}) = A_{11} n_1^2 + A_{22} n_2^2 + A_{33} n_3^2 + \]

\[ 2\left[ (A_{12} + 2A_{66}) - (A_{11} + A_{22}) \right] n_1^2 n_2^2 + 2\left[ (A_{13} + 2A_{55}) - (A_{11} + A_{33}) \right] n_1^2 n_3^2 - 2\left[ (A_{23} + 2A_{44}) - (A_{22} + A_{33}) \right] n_2^2 n_3^2. \]  

(7)

In more compact notation equation (7) may be written as
An expression identical to equation (8) was obtained by Song and Every (2000), however, no mathematical formalism was employed. They took the expression for the $qP$ phase velocity in the $(x_1, x_2)$ plane of a transversely isotropic medium, generalized it by index permutation, and numerically tested their results against the exact expression for the $qP$ phase velocity.

The above equation for the $qP$ phase velocity is indicative of the shape of the slowness surface, which is the inverse of the phase velocity surface (Musgrave, 1970), of the $qP$ in an orthorhombic medium. All terms in $A_{ii}$ ($i=1,2,3$) in Equation 8 have only a second order dependence on $n_i$, while other linearized forms that appear in the literature have a fourth order dependence. This suggests, consistent with the approximation used, that the related slowness surface is an ellipsoid with the lengths of the half axes being $(A_{ii})^{-1/2}$ ($i=1,2,3$). The inclusion of the anellipsoidal deviation terms, $B_{ij}$, $(ij=12,13,23)$, associated with each of the 3 symmetry planes, completes the approximation. This is in agreement with the result that one could expect for anisotropic media with three orthogonal symmetry planes.

The slowness surface associated with equation (8) can only be considered dimensionally correct in velocity if the deviation terms are omitted or set equal to zero. This would yield the degenerate ellipsoidal $qP$ slowness surface, which for weak anisotropy produces a reasonable trend of the actual slowness surface. This “order two only” dependence of the $A_{ii}$ terms will be shown in subsequent sections to also apply to the two quasi-shear modes ($i=4,5,6$).

If mathematically and physically justifiable approximations are made to the exact expression for the $qP$ phase velocity, $B_{ij} = \delta_{ij} - \epsilon_{ij}$ and subsequently $\delta_{ij}$, defined below, differ from those which appeared in earlier literature on a related topic (for example, Tsvankin, 1997). This discrepancy is due to the linearization method used. In the above, for example, the modified (dimensionless) value

$$v_{qP}^2 (n) = A_{11} n_1^2 + A_{22} n_2^2 + A_{33} n_3^2 + 2B_{12} n_1^2 n_2^2 + 2B_{13} n_1^2 n_3^2 + 2B_{23} n_2^2 n_3^2$$  (8)

with the anellipsoidal deviation terms given by

$$B_{12} = (A_{12} + 2A_{66}) - (A_{11} + A_{22})/2$$  (9)
$$B_{13} = (A_{13} + 2A_{55}) - (A_{11} + A_{33})/2$$  (10)
$$B_{23} = (A_{23} + 2A_{44}) - (A_{22} + A_{33})/2.$$  (11)
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\[ \hat{B}_{13} = \frac{(A_{13} + 2A_{55}) - (A_{11} + A_{33})/2}{A_{33}} \]  

(12)

may be compared with the exact dimensionless expression (Gassmann, 1964)

\[ \tilde{B}_{13} = \frac{(A_{13} + A_{55})^2 - (A_{11} - A_{55})(A_{33} - A_{55})}{2A_{33}(A_{33} - A_{55})} \]  

(13)

If this quantity becomes zero, the \( qP \) phase velocity surface in the \((x_1,x_3)\) plane degenerates to an ellipse. As the \( qP - qS_1 \) wave motions are coupled in this plane, the \( qS_1 \) phase velocity surface is forced to be a circle in this plane. It should be restated that equation (12) is the linearized form of the exact deviation term in the \((x_1,x_3)\) plane, given by equation (13).

The related linearized quantity \( \hat{\delta}_{13} \) has the simplified form

\[ \hat{\delta}_{13} = \frac{(A_{13} + 2A_{55} - A_{33})}{A_{33}} \]  

(14)

when compared with the exact expression introduced by Thomsen (1986)

\[ \delta_{13} = \frac{(A_{13} + A_{55})^2 - (A_{33} - A_{55})^2}{2A_{33}(A_{33} - A_{55})} \]  

(15)

Defining the parameter \( \varepsilon_{13} = \frac{(A_{11} - A_{33})}{2A_{33}} \), a measure of ellipticity in the \((x_1,x_3)\) plane, and subtracting this quantity from both Equations 14 and 15 yields

\[ \hat{B}_{13} = \hat{\delta}_{13} - \varepsilon_{13} = \frac{(A_{13} + 2A_{55} - A_{33})}{A_{33}} - \frac{(A_{11} - A_{33})}{2A_{33}} = \left[ \frac{(A_{13} + 2A_{55}) - (A_{11} + A_{33})/2}{A_{33}} \right] \]  

(16)

and

\[ \tilde{B}_{13} = \delta_{13} - \varepsilon_{13} = \frac{(A_{13} + A_{55})^2 - (A_{33} - A_{55})^2}{2A_{33}(A_{33} - A_{55})} - \frac{(A_{11} - A_{33})}{2A_{33}} = \frac{(A_{13} + A_{55})^2 - (A_{11} - A_{55})(A_{33} - A_{55})}{2A_{33}(A_{33} - A_{55})} \]  

(17)

Similar expressions can be derived for both \( \hat{B}_{12} \) and \( \hat{B}_{23} \), and \( \tilde{B}_{12} \) and \( \tilde{B}_{23} \). However, it is more useful in possible related applications to retain the related forms \( B_{13} \), \( B_{12} \) and \( B_{23} \), equations (9) – (11), which are not normalized to any arbitrary quantity.
Returning to equation (8), it should be recalled that the modified quantities $B_y$ are a measure of the deviation of the $qP$ phase velocity from the ellipsoidal. For a linearized weak anisotropy medium, these values should be quite small relative to the $A_i$ terms. It appears to be common practice (Pšenčík and Gajewski, 1998) to require that the terms in $A_i$ $(i = 1, 2, 3)$ are of about the same order of magnitude as the terms involving $B_y$. An isotropic background velocity, $\alpha$, is introduced, essentially putting the $A_i$ terms into their perturbed form, $A_i = A_i^0 + \Delta A_i$, or equivalently, $A_i = \alpha^2 + \Delta A_i$. This is accomplished by adding and subtracting the quantity

$$\alpha^2\left(n_1^2 + n_2^2 + n_3^2\right)$$

(18)

to equation (8), yielding

$$v_{QP}^2(\mathbf{n}) = \alpha^2 + \left(A_{11} - \alpha^2\right)n_1^2 + \left(A_{22} - \alpha^2\right)n_2^2 + \left(A_{33} - \alpha^2\right)n_3^2 + 2B_{12}n_1^2n_2^2 + 2B_{13}n_1^2n_3^2 + 2B_{23}n_2^2n_3^2$$

(19)

or equivalently

$$v_{QP}^2(\mathbf{n}) = \alpha^2\left[1 + \frac{\left(A_{11} - \alpha^2\right)n_1^2}{\alpha^2} + \frac{\left(A_{22} - \alpha^2\right)n_2^2}{\alpha^2} + \frac{\left(A_{33} - \alpha^2\right)n_3^2}{\alpha^2} + \frac{2B_{12}n_1^2n_2^2}{\alpha^2} + \frac{2B_{13}n_1^2n_3^2}{\alpha^2} + \frac{2B_{23}n_2^2n_3^2}{\alpha^2}\right]$$

(20.a)

Taking the square root of both sides of equation (20.a) and expanding the RHS in a binomial series, retaining only the leading terms results in

$$v_{QP}^2(\mathbf{n}) = \alpha\left[1 + \frac{\left(A_{11} - \alpha^2\right)n_1^2}{2\alpha^2} + \frac{\left(A_{22} - \alpha^2\right)n_2^2}{2\alpha^2} + \frac{\left(A_{33} - \alpha^2\right)n_3^2}{2\alpha^2} + \frac{B_{12}n_1^2n_2^2}{\alpha^2} + \frac{B_{13}n_1^2n_3^2}{\alpha^2} + \frac{B_{23}n_2^2n_3^2}{\alpha^2}\right]$$

(20.b)

It is often the case that one of the $\sqrt{A_i}$, $(i = 1, 2, 3)$ is assumed known and consequently the reference velocity, $\alpha$, is set equal to $\sqrt{A_i}$ $(i = 1, 2$ or $3)$, resulting in the loss of one term in equation (20.b).

These last steps in the linearization of the $qP$ phase velocity in an orthorhombic medium, going from equation (8) to (20.b), have been included to maintain a consistency with what appears in the literature. Additionally, the derivation of equation (20b) was thought to be of importance to show the difference that the introduction of equation (5) makes to the final formula; a perturbed ellipsoid, rather than the quartic equation (4) in which the assigning of a specific physical significance to the coefficients is difficult. It is
the linearized approximation given by equation (8) that will be used in a latter section to compare with the exact $qP$ phase velocity in an orthorhombic anisotropic medium.

**LINEARIZED QS$_1$ AND QS$_2$ PHASE VELOCITIES**

The methodology described in Every and Sachse (1992), which will not be repeated here in any detail, involves the application of orthonormal rotation transformation matrices to the Cristoffel matrix $\Gamma$ to isolate either $\hat{\Gamma}_{22}$ or $\hat{\Gamma}_{33}$. This transformation is given by

$$\hat{\Gamma}_{rs} = \Gamma_{pq} a_{rp} a_{sq}$$  \hspace{1cm} (21)

and has been shown to be equivalent to the multiplication of the matrix $\Gamma$ by one of two vectors, $e_2$ or $e_3$. These vectors are required to be orthonormal to the $qP$ phase velocity vector, $n$, which is defined by

$$n = (n_1, n_2, n_3) = e_1 = (e_{11}, e_{12}, e_{13}) = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$$  \hspace{1cm} (22)

The vectors $e_2$ and $e_3$ may be written in terms of the angles $\theta$ and $\phi$, and in terms of the components of the vector $n$ as

$$e_2 = (e_{21}, e_{22}, e_{23}) = (-\cos \theta \cos \phi, -\cos \theta \sin \phi, \sin \theta)$$

$$e_2 = \left( \frac{-n_1 n_3, -n_2 n_3, 1 - n_3^2}{\sqrt{1 - n_3^2}} \right)$$  \hspace{1cm} (23)

$$e_3 = (e_{31}, e_{32}, e_{33}) = (\sin \phi, -\cos \phi, 0)$$

$$e_3 = \left( \frac{n_2, -n_1, 0}{\sqrt{1 - n_3^2}} \right)$$  \hspace{1cm} (24)

The choice of the vectors $e_2$ and $e_3$ is not arbitrary as certain conditions on the orthonormal vector triad

$$(n, e_2, e_3) = (e_1, e_2, e_3)$$  \hspace{1cm} (25)

must be satisfied (Jech and Pšenčík, 1989). Some aspects of this method were dealt with by Vassiliou (1994).

The definition of the $e_i$ ($i = 1, 2, 3$) is given in Figure 1. This schematic indicates that the $qP$ and $qS_1$ phase velocity vectors are constrained to lie within the plane which is perpendicular to the $(x_1, x_2)$ plane and passing through the $x_3$ axis. The result is that the $qS_2$ phase velocity vector is always normal to the plane containing the $qP$ and $qS_1$ phase velocity vectors.
**qS\textsubscript{1} phase velocity in an orthorhombic medium**

For the leading shear wave, \( qS\textsubscript{1} \),

\[
a_{rp} = e_{2p} \\
a_{sq} = e_{2q}
\]

The leading shear wave phase velocity is dependent on the second eigenvalue of the transformed Cristoffel matrix that has been rotated into the elements \( \Gamma_{rs} \). Using the relations between the rotation matrices and the vector \( e_2 \) results in

\[
\Gamma_{22} = \sum_{p,q} \Gamma_{pq} e_{2p} e_{2q}.
\]

Retaining only the first order term, the linearized leading shear wave phase velocity in terms of the density normalized \( A_{ij} \) that have the dimensions of velocity squared is obtained from

\[
\nu_{qS_1}^2 = \Gamma_{22}
\]

Introducing the vector \( e_2 \) results in equation (29) having the form

\[
\nu_{qS_1}^2 = \hat{\Gamma}_{22} = \sum_{p,q} \Gamma_{pq} e_{2p} e_{2q} = \Gamma_{11} e_{21} e_{21} + \Gamma_{22} e_{22} e_{22} + \Gamma_{33} e_{23} e_{23} + 2 \Gamma_{12} e_{21} e_{22} + 2 \Gamma_{13} e_{21} e_{23} + 2 \Gamma_{23} e_{22} e_{23}
\]

The non-zero elements \( \Gamma_{pq} \) of the matrix \( \Gamma \) for an orthorhombic medium may be found in numerous works, for example, Musgrave (1970):

\[
\begin{align*}
\Gamma_{11} &= A_{11} n_1^2 + A_{66} n_2^2 + A_{55} n_3^2 \\
\Gamma_{22} &= A_{66} n_1^2 + A_{22} n_2^2 + A_{44} n_3^2 \\
\Gamma_{33} &= A_{55} n_1^2 + A_{44} n_2^2 + A_{33} n_3^2 \\
\Gamma_{23} &= \Gamma_{32} = (A_{23} + A_{44}) n_3 n_4 \\
\Gamma_{13} &= \Gamma_{31} = (A_{13} + A_{55}) n_1 n_3 \\
\Gamma_{12} &= \Gamma_{21} = (A_{12} + A_{66}) n_1 n_2
\end{align*}
\]

Thus the linearized leading shear wave phase velocity may be written as
\[ v_{qS_5}^2 = \left[ A_{i1}n_1^2 + A_{66}n_2^2 + A_{55}n_3^2 \right] e_{21}e_{21} + \left[ A_{66}n_1^2 + A_{22}n_2^2 + A_{44}n_3^2 \right] e_{22}e_{22} + \\
\left[ A_{55}n_1^2 + A_{44}n_2^2 + A_{33}n_3^2 \right] e_{23}e_{23} + 2 \left( A_{12} + A_{66} \right) n_1n_2e_{21}e_{22} + \\
2 \left( A_{13} + A_{55} \right) n_1n_3e_{21}e_{23} + 2 \left( A_{23} + A_{44} \right) n_2n_3e_{22}e_{23} \tag{37} \]

Adding and subtracting the term
\[
\left[ \left( A_{11} + A_{33} \right) n_1n_3 \right] e_{21}e_{23} + \left[ \left( A_{22} + A_{33} \right) n_2n_3 \right] e_{22}e_{23} + \left[ \left( A_{11} + A_{22} \right) n_1n_2 \right] e_{21}e_{22} - \\
2A_{66}n_1n_2e_{21}e_{22} - 2A_{55}n_1n_3e_{21}e_{23} - 2A_{44}n_2n_3e_{22}e_{23} \tag{38} \]

from equation (37, the following equation results
\[
v_{qS_5}^2 = A_{44} \sin^2 \phi + A_{55} \cos^2 \phi + \\
2 \left[ \left( A_{12} + 2A_{66} \right) - \left( A_{11} + A_{22} \right) / 2 \right] \sin^2 \theta \cos^2 \phi \sin^2 \theta \cos^2 \phi - \\
2 \left[ \left( A_{13} + 2A_{55} \right) - \left( A_{11} + A_{33} \right) / 2 \right] \sin^2 \theta \cos^2 \phi \cos^2 \theta - \\
2 \left[ \left( A_{23} + 2A_{44} \right) - \left( A_{22} + A_{33} \right) / 2 \right] \sin^2 \theta \cos^2 \theta \sin^2 \phi \tag{39.a} \]

or in a more compact form as
\[
v_{qS_5}^2 = A_{44} \sin^2 \phi + A_{55} \cos^2 \phi + 2B_{12} \sin^2 \theta \cos^2 \theta \cos^2 \phi \sin^2 \phi - \\
2B_{13} \sin^2 \theta \cos^2 \theta \cos^2 \phi - 2B_{23} \sin^2 \theta \cos^2 \theta \sin^2 \phi \tag{39.b} \]

where the \( B_{ij} \ (ij = 12, 13, 23) \) were defined in the previous section.

If \( \phi = 0 \) then
\[
v_{qS_5}^2 = A_{55} - 2 \left[ \left( A_{13} + 2A_{55} \right) - \left( A_{11} + A_{33} \right) / 2 \right] \sin^2 \theta \cos^2 \theta \tag{40.a} \]
\[
v_{qS_5}^2 = A_{55} - 2B_{13} \sin^2 \theta \cos^2 \theta \tag{40.b} \]

and for \( \phi = \pi/2 \)
\[
v_{qS_5}^2 = A_{44} - 2 \left[ \left( A_{23} + 2A_{44} \right) - \left( A_{22} + A_{33} \right) / 2 \right] \sin^2 \theta \cos^2 \theta \tag{41.a} \]
\[
v_{qS_5}^2 = A_{44} - 2B_{23} \sin^2 \theta \cos^2 \theta \tag{41.b} \]

In addition, for both \( \theta = 0 \) and \( \theta = \pi/2 \),
\[
v_{qS_5}^2 = A_{44} \sin^2 \phi + A_{55} \cos^2 \phi \tag{42} \]

It is clear from equations (40) – (42) that this mode of quasi–shear propagation is similar to the \( qS_v \) mode in a transversely isotropic medium, at least in the \( (x_1, x_3) \) and \( (x_2, x_3) \) symmetry planes.
As in the $qP$ case, it is at times convenient to have the coefficients of all the terms in the equation for the $qS_1$ phase velocity of a similar magnitude. To accomplish this, add and subtract from equation (39.b) the term

$$\beta^2 \left( \cos^2 \phi + \sin^2 \phi \right)$$

(43)
to yield

$$v_{qS_1}^2 = \beta^2 \left[ 1 + \frac{(A_{44} - \beta^2)}{2\beta^2} \sin^2 \phi + \frac{(A_{55} - \beta^2)}{2\beta^2} \cos^2 \phi + \frac{2B_{12} \sin^2 \theta \cos^2 \phi \sin^2 \phi}{\beta^2} - \frac{2B_{13} \sin^2 \theta \cos^2 \phi \sin^2 \phi}{\beta^2} \right]$$

(44)

The linearization process may be completed by taking the square root of both sides of equation (44) and expanding the RHS in a binomial series, retaining only the leading terms, to obtain

$$v_{qS_1} = \beta \left[ 1 + \frac{(A_{44} - \beta^2)}{2\beta^2} \sin^2 \phi + \frac{(A_{55} - \beta^2)}{2\beta^2} \cos^2 \phi + \frac{B_{12} \sin^2 \theta \cos^2 \phi \sin^2 \phi}{\beta^2} - \frac{B_{13} \sin^2 \theta \cos^2 \phi \sin^2 \phi}{\beta^2} \right]$$

(45)

It is evident from the above derivation that the linearized $qP$ phase velocity may be obtained in a similar manner. This would involve isolating $\Gamma_{11}$ using rotation matrices associated with the vector $n = e_1$, resulting in $v_{qP}^2 = \tilde{\Gamma}_{11}$.

**$qS_2$ phase velocity in an orthorhombic medium**

From the introduction in the previous section on the $qS_1$ phase velocity it may be determined (Every and Sachse, 1992) that the expression for the second shear wave phase velocity, $v_{qS_2}$, in an orthorhombic medium is related to the matrix element $\tilde{\Gamma}_{33}$ in the same manner as $v_{qS_1}$ is related to $\tilde{\Gamma}_{11}$. From equations (21), (26) and (27) in the previous section the following relation, in terms of the vector $e_3$, is obtained
Orthorhombic linearized phase velocities

\[ \hat{\Gamma}_{33} = \sum_{p,q} \Gamma_{pq} e_p e_{3q} \quad (46) \]

Thus the linearized expression for the second quasi-shear phase velocity may be written as

\[ \nu_{qS_2}^2 = \hat{\Gamma}_{33} = \sum_{p,q} \Gamma_{pq} e_p e_{3q} \]

\[ = \Gamma_{11} e_{31} e_{31} + \Gamma_{22} e_{32} e_{32} + 2\Gamma_{12} e_{31} e_{32} \quad (47) \]

Introducing the expressions for the \( \Gamma_{pq} \) defined in equations (31) – (36) yields

\[ \nu_{qS_2}^2 = \left[ A_{11} n_1^2 + A_{66} n_2^2 + A_{55} n_3^2 \right] e_{31} e_{31} + \left[ A_{66} n_1^2 + A_{22} n_2^2 + A_{44} n_3^2 \right] e_{32} e_{32} + 2 \left( A_{12} + A_{66} \right) n_1 n_2 e_{31} e_{32} \quad (48) \]

In this case, adding and subtracting the terms

\[ \left( A_{11} + A_{22} \right) n_1 n_2 e_{31} e_{32} - 2 A_{66} n_1 n_2 e_{31} e_{32} \quad (49) \]

from equation (48) results in the phase velocity for the second quasi-shear wave being given as

\[ \nu_{qS_2}^2 = A_{44} \cos^2 \theta \cos^2 \phi + A_{55} \cos^2 \theta \sin^2 \phi + A_{66} \sin^2 \theta - 2 \left[ \left( A_{12} + 2 A_{66} \right) - \left( A_{11} + A_{22} \right) / 2 \right] \sin^2 \theta \sin^2 \phi \cos^2 \phi \quad (50.a) \]

or equivalently, with \( B_{12} \) defined by equation (9),

\[ \nu_{qS_2}^2 = A_{44} \cos^2 \theta \cos^2 \phi + A_{55} \cos^2 \theta \sin^2 \phi + A_{66} \sin^2 \theta - 2 B_{12} \sin^2 \theta \sin^2 \phi \cos^2 \phi \quad (50.b) \]

If \( \phi = 0 \), then equation (50.b) reduces to

\[ \nu_{qS_2}^2 = A_{44} \cos^2 \theta + A_{66} \sin^2 \theta \quad (51) \]

and, if \( \phi = \pi/2 \),

\[ \nu_{qS_2}^2 = A_{55} \cos^2 \theta + A_{66} \sin^2 \theta \quad (52) \]

while for \( \theta = 0 \)

\[ \nu_{qS_2}^2 = A_{44} \cos^2 \phi + A_{55} \sin^2 \phi \quad (53) \]

and for \( \theta = \pi/2 \)
The preceding four equations indicate that this quasi – shear mode of wave propagation behaves like the $qS_\mu$ mode in a transversely isotropic medium, again, in the $(x_1, x_3)$ and $(x_2, x_3)$ symmetry planes. The behaviour in the $(x_1, x_2)$ plane cannot be placed in any category.

In a manner similar to the previous two sections, the equalization of magnitudes of the coefficients of equation (50.b) is once again achieved by adding and subtracting the term 

$$\beta^2 \left[ \cos^2 \theta \cos^2 \phi + \cos^2 \theta \sin^2 \phi + \sin^2 \theta \right]$$

from equation (50.b) to obtain

$$v^2_{qS_\mu} = \beta^2 \left[ 1 + \frac{(A_{44} - \beta^2)}{\beta^2} \cos^2 \theta \cos^2 \phi + \frac{(A_{55} - \beta^2)}{\beta^2} \cos^2 \theta \sin^2 \phi + \frac{(A_{66} - \beta^2)}{\beta^2} \sin^2 \theta - \frac{2B_{12} \sin^2 \phi \cos^2 \phi}{\beta^2} \right]$$

(56)

The linearization process requires taking the square root of equation (56) and expanding the RHS of the resulting equation in a binomial series, again retaining only the leading terms, to finally obtain

$$v^2_{qS_\mu} \approx \beta^2 \left[ 1 + \frac{(A_{44} - \beta^2)}{2\beta^2} \cos^2 \theta \cos^2 \phi + \frac{(A_{55} - \beta^2)}{2\beta^2} \cos^2 \theta \sin^2 \phi + \frac{(A_{66} - \beta^2)}{2\beta^2} \sin^2 \theta - \frac{B_{12} \sin^2 \phi \cos^2 \phi}{\beta^2} \right]$$

(57)

It has been convenient in the last shear wave cases to retain the $(\theta, \phi)$ notation rather than the $(e_{ij}, e_{ij})$ notation as it provides more clarity in identifying the wave’s behaviour.

As approximate (linearized) expressions have now been obtained for the three modes of elastic waves propagating in an orthorhombic anisotropic medium, a check on their accuracy will be performed using the exact expressions given by Every (1980) or Schoenberg and Helbig (1997). It should be noted that in deriving the exact expressions by solving a cubic equation with 3 real roots, the two shear modes for which exact expressions are given in the works cited above, are correctly named the "fast" and "slow" shear wave modes. There is not necessarily a one to one correspondence to the $qS_1$ and $qS_2$ modes discussed above. This is due to the fact that the $qS_1$ and $qS_2$ shear wavefront surfaces may intersect one another, while the "fast" shear mode in the exact formulation is always that, the fastest of the two shear modes, and the "slow" shear mode is similar. Kiss singularities occur when these two wave surfaces are about to
intersect, resulting in some of the related formulae describing the two wave types becoming undefined at these points.

**NUMERICAL RESULTS**

The model used is a fairly significant modification of the example given in Pšenčík and Gajewski (1998), with the 21 parameter model downgraded to an orthorhombic input data set. It is defined in terms of the density normalized \((velocity)^2\) parameters, \(A_{ij}\), with dimensions of \(km^2/s^2\) as

\[
\begin{bmatrix}
19.3 & 0.9 & 1.3 & 0 & 0 & 0 \\
17.4 & 0.2 & 0 & 0 & 0 & 0 \\
14.1 & 0 & 0 & 0 & 0 & 0 \\
5.1 & 0 & 0 & 0 & 0 & 0 \\
5.5 & 0 & 0 & 0 & 0 & 0 \\
4.6 & & & & & \\
\end{bmatrix}
\]

(58)

To put the \(B_{ij}\) \((ij=12,13,23)\) in perspective with the \(A_{ij}\), the following numerical values obtained from equations (9) – (11) follow

\[
B_{12} = (A_{12} + 2A_{66}) - (A_{11} + A_{22})/2 = -8.25 \text{ km}^2/\text{s}^2 \quad (-0.585) \quad (9')
\]

\[
B_{13} = (A_{13} + 2A_{55}) - (A_{11} + A_{33})/2 = -4.4 \text{ km}^2/\text{s}^2 \quad (-0.312) \quad (10')
\]

\[
B_{23} = (A_{23} + 2A_{44}) - (A_{22} + A_{33})/2 = -5.35 \text{ km}^2/\text{s}^2 \quad (-0.379) \quad (11')
\]

The bracketed values are the \(B_{ij}\) made dimensionless by dividing each by \(A_{33}\).

The exact values of the \(qP\), \(qS_1\) and \(qS_2\) phase velocities are computed using equations from Every (1980) or Schoenberg and Helbig (1997). The exact (red) results, together with the approximate velocities in black, derived in the text for the \(qP\) phase velocity and given by Equation 18 are shown in Figures 2 and 4.

The exact (red) and approximate (black) \(qS_1\) and \(qS_2\) phase velocities obtained from equations (39.b) and (50.b) are presented in Figures 3 and 5 for the same model used for the \(qP\) case. The differences between the exact and approximate phase velocities are much more evident in Figures 1 and 2 due to the scale used on the vertical axis of the plots.

**CONCLUSIONS**

Using the linearized expressions for the \(qP\), \(qS_1\) and \(qS_2\) phase velocities in an orthorhombic anisotropic medium as a starting point, alterations to the formulae were made to put them in a more indicative form. All \((A_{ij}, i=1,2,\ldots,6)\) terms in the phase velocities are of order 2 in the phase velocity vector components rather than order 4,
which the standard linearization procedure produces. The expressions for the phase velocities are all of a form such that they are ellipsoids with anellipsoidal correction terms, which are of order 4 in the phase velocity vector components.

Compared with other expressions for these three phase velocities, which appear in numerous works in the literature, almost without exception in terms of \((\varepsilon_\mu, \mu = x, y, z)\) and \((\delta_\mu, \mu = x, y, z)\), the formulae presented here are much less cumbersome. All quantities within the formulae can be associated with a physically measurable quantity. The results presented display the close match with the exact formulae.

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FIG. 1. Geometry of the phase velocity vectors used in the text.
FIG. 2. qP phase velocities – red is exact and black is approximate. Azimuthal angles of 0, $\pi/4$ and $\pi/2$ radians. Vertical scale enhances any mismatch.
FIG. 3. qS\textsubscript{1} and qS\textsubscript{2} phase velocities – red is exact and black is approximate. Azimuthal angles of 0, \(\pi/4\) and \(\pi/2\). Vertical scale enhances any mismatch.
FIG. 4. Polar plot of the same data used to produce the panels in Figure 1.
FIG. 5. Polar plot of the same data used to produce the panels in Figure 2.