Minimum-phase revisited

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ABSTRACT

The assumption of a minimum-phase seismic wavelet is central to the Wiener and Gabor deconvolution algorithms. The concept of minimum phase is well understood by the geophysics community for the case of digital signals. The extension of minimum phase to analog signals, however, has not received sufficient attention in the literature. Certain well-known properties about digital minimum phase signals, such as convolutional invertibility, stability, and Hilbert transform relationships between Fourier amplitude and phase spectra, do not carry over so easily to the continuous setting.

We have found it necessary to develop the analog theory of minimum phase within the rather general framework of tempered distributions. Subsequently, the theory allows us to extend the concept of a minimum-phase filter (linear operator) to the general setting of nonstationary analog systems. Such a filter is used explicitly in theoretical development of Gabor deconvolution.

INTRODUCTION

The mathematical topics considered in this paper fall under the broad heading of functional analysis. The books by Rudin (1973) Conway (1990), Reed and Simon (1980), and Cheney (2001) are all excellent resources. These texts generally assume an elementary background in real analysis (see Rudin, 1976, or Royden, 1988) and some experience with complex analysis (see Lang, 1977, or Churchill and Brown, 1984). Halmos' book (1967) has many interesting problems on Hilbert space, complete with hints and solutions. The two volumes by Dunford and Schwartz (1958), are the classic treatises on linear operator theory. The English translation of the Russian series on generalized functions (i.e., distributions) by Gel'fand, et al. (1964–1968) is outstanding.

This paper is presented in two main parts. The first part fixes some notation, and serves as an introduction to various function spaces used for representing signals and images. It also contains a discussion on the space of tempered distributions, a prerequisite to understanding the concept of minimum phase, and more generally, the theory of inverse problems in imaging and signal analysis.

The second part develops the concept of minimum phase. This includes the derivation of the Hilbert transform from the assumption of causality, and its subsequent extension to the space of tempered distributions. The concept of minimum phase is reasonably well understood for the discrete case (e.g., Robinson, 1967). The continuous case, in our opinion, has not received sufficiently careful attention in the literature. There are some commonly held misconceptions surrounding the theory, and it is the main objective of this paper to overcome these issues by presenting a logically consistent and mathematically rigorous

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theory of minimum phase.

The stage is then set for the development of the theory of nonstationary, minimum-phase filters and their inverses. These are the pseudodifferential (and Fourier integral) operators which preserve minimum phase; that is, the ones which carry minimum-phase signals to minimum-phase signals.

FUNCTION SPACES FOR REPRESENTING SIGNALS AND IMAGES

A signal is a real-valued function of a single time variable, while an image is a real-valued function of up to three spatial variables. By considering classes of functions defined on $d-$dimensional Euclidean space, $\mathbf{R}^d$, the many properties that signals and images have in common can be studied simultaneously. For this reason and for brevity, we adopt a common convention and often refer to both signals and images as signals. Similarly, much of what is called "time-frequency analysis" (e.g. Gröchenig, 2001) is carried out on functions of d variables, and so its scope encompasses both signals and images.

Unless otherwise stated, by a function we shall mean a mapping $f : \mathbf{R}^d \to \mathbf{C}$ from real d-dimensional Euclidean space to the set of complex numbers. The support of a function f, denoted by Supp(f), is the closure in $\mathbf{R}^d$ of the set of points at which f is nonzero.

We will also consider functions $f : \mathbf{C} \to \mathbf{C}$ of a complex variable (see above, and Ahlfors, 1979, and Paley and Wiener, 1967, for more advanced results). Such a function is analytic at a point z if it is complex-differentiable at that point. If f is analytic at every point $z \in \mathbf{C}$, then f is called entire. A function of a complex variable has the remarkable property of being infinitely complex differentiable at each point at which it is analytic.

The Hilbert space of finite-energy signals

The functions we shall consider for representing signals and images will have finite, measurable energy. These belong to the class of the so-called $\mathcal{L}^2$-functions, a space of complex-valued, measurable (see e.g., Rudin, 1976, or Royden, 1988, for more on measurability) functions on $\mathbf{R}^d$, given by

$$\mathcal{L}^2(\mathbf{R}^d) = \{f : \mathbf{R}^d \to \mathbf{C} : \|f\|_2 < \infty\}, \tag{1}$$

where the energy, or norm, of f is defined by

$$\|f\|_2 = \left(\int_{\mathbf{R}^d} |f(x)|^2 \, dx \right)^{1/2}. \tag{2}$$

The reason for the subscript " 2 " in the notation for the norm will become clear in the next subsection. An inner product is defined on $\mathcal{L}^2(\mathbf{R}^d)$ by

$$\langle f, g \rangle = \int_{\mathbf{R}^d} f(x)\overline{g(x)} \, dx. \tag{3}$$

It induces the norm on $\mathcal{L}^2(\mathbf{R}^d)$, since $\|f\|_2 = \sqrt{\langle f, f \rangle}$. The inner product space $\mathcal{L}^2(\mathbf{R}^d)$ is complete (all Cauchy sequences converge within the space) with respect to the distance function induced by the norm,

$$\operatorname{dist}(f, g) = \|f - g\|_2, \tag{4}$$

and thus it forms a Hilbert space (that is, a vector space with an inner product, which is also complete with respect to the distance function induced by the inner product). Indeed, $\mathcal{L}^2(\mathbf{R}^d)$ is the prototypical infinite-dimensional Hilbert space. It is perhaps the most natural infinite-dimensional generalization of the Euclidean spaces, which are finite-dimensional Hilbert spaces. A sequence of vectors $f_n \in \mathcal{L}^2(\mathbf{R}^d)$ is a Cauchy sequence if $\|f_m - f_n\|_2 \to 0$ as $m, n \to \infty$. Completeness means that the sequence converges in norm to a unique function $f \in \mathcal{L}^2(\mathbf{R}^d)$: $\|f - f_n\|_2 \to 0$ as $n \to \infty$.

Although a realizable signal is real-valued, its Fourier transform is typically complex-valued. Also, it is well known that the Fourier transform maps $\mathcal{L}^2(\mathbf{R}^d)$ onto itself, which motivates us to consider signals in this more general setting.

The Lebesgue spaces

For $1 \leq p < \infty$, a measurable function f (see e.g., Rudin, 1976, or Royden, 1988 for more on measurability) is in $\mathcal{L}^p(\mathbf{R}^d)$ if its p-norm is finite; that is, if

$$\|f\|_p = \left(\int_{\mathbf{R}^d} |f(x)|^p \, dx \right)^{1/p} < \infty. \tag{5}$$

Thus, for $1 \leq p < \infty$, a measurable function f is in $\mathcal{L}^p(\mathbf{R}^d)$ if and only if $|f|^p \in \mathcal{L}^1(\mathbf{R}^d)$. For $p = \infty$, $f \in \mathcal{L}^\infty(\mathbf{R}^d)$ if f is measurable and

$$\|f\|_\infty = \text{ess sup}|f(x)| < \infty. \tag{6}$$

Here, “ess sup" means “essential supremum" (Rudin, 1976, Lang, 1969, or Royden, 1988, for details). For continuous functions, this is the same as the supremum.

The most interesting $\mathcal{L}^p-$spaces for our purposes are for $p = 1, 2,$ or ∞. Elements of $\mathcal{L}^p(\mathbf{R}^d)$ are actually equivalence classes of measurable functions which are identified if and only if they agree “almost everywhere", that is, except on a set of zero measure. For example, the zero function can be perturbed at countably many points, and still remain in the same equivalence class. Generally, two measurable functions f and g are equivalent if and only if $f - g = 0$ almost everywhere. It is customary to use the same notation both for a function and the equivalence class it represents.

The Schwartz space of rapidly decreasing signals

The $\mathcal{L}^p$-spaces contain a host of rather exotic functions, lacking basic regularity properties such as differentiability or continuity. Even so, there exists a space of C^∞ (smooth, or infinitely differentiable) rapidly-decreasing functions, called the Schwartz class, which is dense in $\mathcal{L}^p$ for $1 \leq p < \infty$, (and weak-* dense in $\mathcal{L}^\infty$, but we shall not need this fact) (see Conway, 1990, or Rudin, 1973). This means that we can approximate an $\mathcal{L}^p$-function arbitrarily well by a smooth function in the $\mathcal{L}^p$-norm.

A complex-valued function $\phi(x)$ on $\mathbf{R}^d$ is rapidly decreasing if it is infinitely differentiable and it and each of its derivatives decays faster than the inverse of any polynomial; that is, if for all multiindices α, β, the functions $x^\alpha \partial^\beta \phi(x)$ are bounded (Saint Raymond, 1991). A multiindex, α, is a d-tuple of positive integers, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathbf{Z}_+^d$. Thus, with $x = (x_1, x_2, \ldots, x_d) \in \mathbf{R}^d$, the polynomial x^α is defined as

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_d^{\alpha_d}, \tag{7}$$

and ∂^α means

$$\partial^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_d^{\alpha_d}, \tag{8}$$

where $\partial_k^{\alpha_k}$ is the ${\alpha_k}^{th}$ partial derivative with respect to the k^{th} coordinate x_k.

The set of all rapidly decreasing functions is called Schwartz space and is denoted by $\mathcal{S}(\mathbf{R}^d)$. Examples of Schwartz functions include Gaussians and the so-called "bump" functions.

An important feature of $\mathcal{S}(\mathbf{R}^d)$ is that the Fourier transform (formula (10) below) gives a one-to-one correspondence between $\mathcal{S}(\mathbf{R}^d)$ and itself, with the operation of pointwise multiplication being carried over into convolution and vice versa (see Conway (1990) or Rudin (1973)).

By Plancherel's theorem,

$$\|f\|_2 = \left(\frac{1}{2\pi}\right)^{\frac{d}{2}} \|\hat{f}\|_2 \tag{9}$$

for all $f \in \mathcal{S}(\mathbf{R}^d)$, and since $\mathcal{S}(\mathbf{R}^d)$ is dense in $\mathcal{L}^2(\mathbf{R}^d)$, the Fourier transform extends to a unitary operator on $\mathcal{L}^2(\mathbf{R}^d)$. We emphasize that the formula

$$\hat{f}(\xi) = \int_{\mathbf{R}^d} f(x) e^{-i\xi \cdot x}\, dx \tag{10}$$

for the Fourier transform of f only makes sense when f is in $\mathcal{L}^1(\mathbf{R}^d)$—otherwise the integral is not well-defined. The point is that although the Fourier transform extends to an isometry on $\mathcal{L}^2(\mathbf{R}^d)$, its action on $\mathcal{L}^2$-functions outside $\mathcal{L}^1(\mathbf{R}^d)$ is not given by formula (10).

An essential property of a function $f \in \mathcal{L}^1(\mathbf{R}^d)$ is that its Fourier transform is a continuous function that vanishes at infinity: $|\hat{f}(\xi)| \to 0$ as $|\xi| \to \infty$ (see Conway, 1990, or Rudin, 1973)). This decay property is not obvious by inspection of 10 — the proof involves the density of $\mathcal{S}(\mathbf{R}^d)$ in $\mathcal{L}^1(\mathbf{R}^d)$, and the completeness of $C_o(\mathbf{R}^d)$ (the space of continuous functions that vanish at infinity) in the $\mathcal{L}^\infty-$norm (see Rudin, 1973, Theorem 7.5).

The space of tempered distributions

Distributions, or generalized functions, arise as a generalization of the idea of a bounded function. Motivated by the desire to understand Dirac's delta-function, Laurent Schwartz developed his theory of distributions in 1944—and received the Fields Medal in 1950 for this work (see, e.g. Schwartz, 1957). Schwartz actually studied generalized functions later than many others (e.g., Hadamard, Riesz, Sobolev, Bochner), but he was the first to systematize the theory, while relating all the earlier approaches and establishing many important results (Gel'fand and Shilov, 1964).

It would be difficult, if not impossible, to avoid this subject in any rigorous account of inverse-filtering theory. For an authoritative account of generalized function theory and its applications, the English translation of the Russian 5-volume series, *Generalized Functions*, by Gel'fand, et al., is highly recommended (Gel'fand, et al., 1966, through to Gel'fand and Vilenkin, 1964). See also Lang (1969), Cheney (2001), and Rudin (1973).

Definition 0.1 *A tempered distribution T is a continuous linear functional on Schwartz space; that is, a continuous linear mapping $T : \mathcal{S}(\mathbf{R}^d) \to \mathbf{C}$.*

The continuity is with respect to the family of norms (Saint Raymond, 1991):

$$|\phi|_k = \sup\{|x^\alpha \partial^\beta \phi(x)| : x \in \mathbf{R}^d \text{ and } |\alpha + \beta| \leq k\}, \tag{11}$$

indexed by the positive integers, $k \in \mathbf{Z}_+$, and where α and β in $\mathbf{Z}_+^d$ are multiindices. Given a multiindex α, the notation $|\alpha|$ means the sum $|\alpha| = \alpha_1 + \alpha_2 + \cdots \alpha_d$.

The space of tempered distributions is denoted by $\mathcal{S}'(\mathbf{R}^d)$. A very large, and possibly the most important class of tempered distributions comes from the tempered functions on $\mathbf{R}^d$ (see Wong, 1998). A measurable function f on $\mathbf{R}^d$ is said to be tempered if

$$\int_{\mathbf{R}^d} \frac{|f(x)|}{(1 + |x|)^N}\, dx < \infty \tag{12}$$

for some positive integer N. Convergence of the integral (12) implies f has at most polynomial growth:

$$|f(x)| \leq C(1 + |x|)^N. \tag{13}$$

Given any tempered function f on $\mathbf{R}^d$, the linear functional T_f on $\mathcal{S}(\mathbf{R}^d)$ given as

$$\int_{\mathbf{R}^d} f(x)\phi(x)\, dx,\ \phi \in \mathcal{S}(\mathbf{R}^d) \tag{14}$$

is a tempered distribution. Evidently, every $\mathcal{L}^p-$function is tempered, ($1 \leq p \leq \infty$), and so (14) defines a tempered distribution for any $f \in \mathcal{L}^p(\mathbf{R}^d)$.

Definition 0.2 *The support of a distribution T, denoted by Supp (T), is the smallest closed set $F \subseteq \mathbf{R}^d$ satisfying the following property:*

$$T(\phi) = 0 \textit{ for all } \phi \in \mathcal{S}(\mathbf{R}^d) \textit{ with Supp } (\phi) \subseteq \mathbf{R}^d \setminus F. \tag{15}$$

If Γ is the collection of all closed sets $F \subseteq \mathbf{R}^d$ satisfying property (15), then (Cheney, 2001)

$$\text{Supp } (\phi) = \bigcap \{F : F \in \Gamma\}. \tag{16}$$

The definition of support for distributions is compatible with the definition of support for functions, in the sense that if T_g is the tempered distribution corresponding to a suitably behaved function g (e.g., any tempered function), then its support agrees with the usual definition of support for a function (see Cheney, 2001). This is why the same notation is conventionally used for both definitions.

Distributional derivatives

The concept of differentiation of functions extends to the entire class of tempered distributions, as follows.

Definition 0.3 *(e.g., Lang, 1969; Saint Raymond, 1991) Let* $D^\alpha = (-i)^{|\alpha|}\partial^\alpha$, $(\alpha \in \mathbf{Z}_+^d)$ *and let* $T \in \mathcal{S}(\mathbf{R}^d)$. *Then the derivative of* T *is defined as*

$$D^\alpha T = T D^\alpha. \tag{17}$$

The use of the differential operator $D^\alpha = (-i)^{|\alpha|}\partial^\alpha$ helps to simplify expressions involving partial derivatives (D is then a self-adjoint operator).

It is a remarkable fact that any tempered distribution supported at the origin can be expressed as a finite linear combination of derivatives of the Dirac distribution (Lang, 1969, Theorem 5, p. 387). More generally, any tempered distribution with discrete (consisting of isolated points) support is just a countable superposition of translates of distributions supported at the origin (Lang, 1969). One can also show (e.g., Lang, 1969) that a distribution with compact (closed and bounded) support is $\mathcal{D}T_f$, for some differential operator $\mathcal{D}$ and continuous function f.

The Fourier transform of a distribution

Definition 0.4 *The Fourier transform,* $\hat{T}$, *of a tempered distribution* $T \in \mathcal{S}'(\mathbf{R}^d)$ *is defined by its action on Schwartz functions, (see Lang, 1969; Cheney, 2001; or Rudin, 1973) as*

$$\hat{T}(\phi) = T(\hat{\phi}). \tag{18}$$

An important property of the class of tempered distributions is the following.

Proposition 0.5 *The Fourier transform is a continuous linear bijection of* $\mathcal{S}'(\boldsymbol{R}^d)$ *onto itself.*

Proof The Fourier transform $\mathcal{F}$ is a continuous linear bijection of $\mathcal{S}(\mathbf{R}^d)$ onto itself. Since the Fourier transform $\hat{T}$ of a tempered distribution T is defined by a composition of continuous linear maps, namely $\hat{T} = T \circ \mathcal{F}$, the mapping $T \mapsto \hat{T}$ is continuous and linear. In particular, $\hat{T} \in \mathcal{S}'(\mathbf{R}^d)$, and so the Fourier transform maps $\mathcal{S}'(\mathbf{R}^d)$ into itself. Given any $T \in \mathcal{S}'(\mathbf{R}^d)$, put $S = T \circ \mathcal{F}^3 \in \mathcal{S}'(\mathbf{R}^d)$. Then since $\mathcal{F}^4$ is the identity mapping on $\mathcal{S}'(\mathbf{R}^d)$, $T = \hat{S}$, and the Fourier transform maps $\mathcal{S}'(\mathbf{R}^d)$ onto itself. It is one to one since if $\hat{T} = \hat{S}$, then the Fourier transform applied to both sides three times gives back $T = S$. ∎

This concludes our overview of the key concepts we shall need from Functional Analysis. We now turn to the main development of the theory surrounding the concept of minimum phase.

CAUSALITY AND MINIMUM PHASE

Causality and the Hilbert transform

Definition 0.6 *A tempered distribution (or function) T is causal if Supp$(T) \subseteq [0, \infty)$.*

We shall make use of the Hilbert transform, $\mathcal{H}$, a bounded linear operator on $\mathcal{L}^2(\mathbf{R})$. We first define its action on Schwartz functions as

$$(\mathcal{H}s)(\omega) = \text{ P.V. } \frac{1}{\pi} \int_{\mathbf{R}} \frac{s}{\omega - \xi} \, d\xi = \frac{1}{\pi} \lim_{R \to \infty} \lim_{\epsilon \to 0} \left[\int_{-R}^{-\epsilon} \frac{s}{\omega - \xi} \, d\xi + \int_{\epsilon}^{R} \frac{s}{\omega - \xi} \, d\xi \right], \quad (19)$$

where P.V. denotes the Cauchy principal value of the integral (see e.g., Ahlfors, 1979; Lang, 1977). As we shall see, the Hilbert transform extends as a bounded linear operator on $\mathcal{L}^2(\mathbf{R})$, and satisfies the following properties on $\mathcal{L}^2(\mathbf{R})$: $\mathcal{H}^2 = -I$, $\mathcal{H}^* = -\mathcal{H}$, and thus $\mathcal{H}^{-1} = \mathcal{H}^*$; that is, $\mathcal{H}$ is unitary. Also, if s is real-valued then so is $\mathcal{H}s$. Some authors (e.g., Connes, 1994) have a factor of $1/i$ in the definition of $\mathcal{H}$, which makes $\mathcal{H}$ a self-adjoint unitary.

Proposition 0.7 *If s is a causal Schwartz function, then the real and imaginary parts of its Fourier transform are a Hilbert transform pair, namely*

$$\Re\left\{\hat{s}(\omega)\right\} = (\mathcal{H}\,\Im\{\hat{s}\})(\omega), \quad (20)$$

where $\Re$ and $\Im$ are used to denote the real and imaginary parts, respectively.

Proof Suppose $s \in \mathcal{S}(\mathbf{R})$ is causal. Then $s = hs$, where h is the Heaviside function; that is, h is the characteristic function of $[0, \infty)$. Now the Fourier transform of h is given by (e.g., Papoulis, 1977, or Pujol, 2003)

$$\hat{h}(\omega) = \pi\delta(\omega) - \frac{i}{\omega}, \quad (21)$$

and so

$$\hat{s}(\omega) = \left(\hat{s} * \hat{h}\right)(\omega) = \frac{1}{2}\left(\hat{s}(\omega) - \frac{i}{\pi}\int_{\mathbf{R}} \frac{\hat{s}(\xi)}{\omega - \xi}\, d\xi\right). \quad (22)$$

Thus

$$\hat{s}(\omega) = -\frac{i}{\pi}\int_{\mathbf{R}} \frac{\hat{s}(\xi)}{\omega - \xi}\, d\xi, \quad (23)$$

that is,

$$\Re\left\{\hat{s}(\omega)\right\} = \Im\left\{\frac{1}{\pi}\int_{\mathbf{R}} \frac{\hat{s}(\xi)}{\omega - \xi}\, d\xi\right\} = \frac{1}{\pi}\int_{\mathbf{R}} \frac{\Im\{\hat{s}(\xi)\}}{\omega - \xi}\, d\xi = (\mathcal{H}\,\Im\{\hat{s}\})(\omega), \quad (24)$$

which is what we wanted to show. ∎

Expression (23) in the proof shows that the action of the Hilbert transform on the spectrum of a causal Schwartz function is rotation by $90°$ in the complex plane: $\mathcal{H}\widehat{s} = i\widehat{s}$. From the relationship $\widehat{\widetilde{s}} = \overline{\widehat{s}}$, where $\widetilde{s}(t) = s(-t)$ is the time reversal of s, we see that the Hilbert transform rotates the spectrum of an anti-causal signal by $-90°$.

Given any $f \in \mathcal{L}^2(\mathbf{R})$, f can be represented uniquely (up to a set of measure zero, hence uniquely as an $\mathcal{L}^2$-function) in the form

$$f = f_+ + f_-\,, \tag{25}$$

where f_+ and f_- are the causal and anti-causal parts of f, respectively. That is, $\mathcal{L}^2(\mathbf{R})$ can be identified with the direct sum

$$\mathcal{L}^2(\mathbf{R}) = \mathcal{L}^2(\mathbf{R})_+ \oplus \mathcal{L}^2(\mathbf{R})_-\,, \tag{26}$$

where $\mathcal{L}^2(\mathbf{R})_+$ and $\mathcal{L}^2(\mathbf{R})_-$ are the orthogonal subspaces of $\mathcal{L}^2(\mathbf{R})$ consisting of the causal and anti-causal $\mathcal{L}^2$-functions, respectively. Applying the Fourier transform, we can then identify $\mathcal{F}\mathcal{L}^2(\mathbf{R})$ with the direct sum

$$\mathcal{F}\mathcal{L}^2(\mathbf{R}) \cong \mathcal{F}\mathcal{L}^2(\mathbf{R})_+ \oplus \mathcal{F}\mathcal{L}^2(\mathbf{R})_-\,; \tag{27}$$

and similarly for Schwartz space:

$$\mathcal{F}\mathcal{S}(\mathbf{R}) \cong \mathcal{F}\mathcal{S}(\mathbf{R})_+ \oplus \mathcal{F}\mathcal{S}(\mathbf{R})_-. \tag{28}$$

Knowing how the Hilbert transform acts on the spectra of causal and anti-causal Schwartz functions, we see that $\mathcal{H}$ acts diagonally on $\mathcal{F}\mathcal{S}(\mathbf{R})$:

$$\mathcal{H}(\hat{f}) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} \widehat{f_+} \\ \widehat{f_-} \end{pmatrix} = i \begin{pmatrix} \widehat{f_+} \\ -\widehat{f_-} \end{pmatrix}. \tag{29}$$

Evidently, expression (29) defines a continuous linear mapping $\mathcal{H} : \mathcal{S}(\mathbf{R}) \to \mathcal{S}(\mathbf{R})$. Since $\mathcal{S}(\mathbf{R})$ is dense in $\mathcal{L}^2(\mathbf{R})$, and $\mathcal{F}\mathcal{S}(\mathbf{R})_\pm \subset \mathcal{F}\mathcal{L}^2(\mathbf{R})_\pm$, we have the following result.

Proposition 0.8 *The action defined by expression (29) extends the Hilbert transform to a unitary operator on $\mathcal{L}^2(\mathbf{R})$. The eigenvalues of the Hilbert transform are exactly i and $-i$, and the corresponding eigenspaces are the Fourier spectra of the causal and anti-causal signals, respectively.*

Actually, the same definition extends the domain of the Hilbert transform without difficulty to the space of tempered distributions.

The fact that $\mathcal{H}$ is unitary is easily seen as follows:

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}^* = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{30}$$

The following property of the causal $\mathcal{L}^1$-functions in terms of their Fourier spectra is used in the next subsection, where we discuss the topic of minimum phase.

Theorem 0.9 *Let $f \in \mathcal{L}^1(\mathbf{R})$. If f is causal, then its Fourier transform is analytic in the open lower half of the complex plane.*

Proof Let $f \in \mathcal{L}^1(\mathbf{R})$. Then its Fourier transform is computable using formula (10). Writing $\xi = u + iv$, with $u, v \in \mathbf{R}$, we have

$$\hat{f}(\xi) = \int_{\mathbf{R}} f(x)e^{-i\xi \cdot x}\,dx = \int_{\mathbf{R}} f(x)e^{v\cdot x}e^{-iu\cdot x}\,dx, \tag{31}$$

which, for each fixed v where the integral converges, is the Fourier transform of $f(x)e^{v\cdot x}$, evaluated at u. If f is causal, then for $v \leq 0$,

$$\left| f(x)e^{v\cdot x} \right| \leq \left| f(x) \right|, \tag{32}$$

and the integral converges. In fact, if f is causal and $v < 0$, then for any $N \in \mathbf{N}$, the function $(-ix)^N f(x)e^{v\cdot x}$ is absolutely integrable: this follows from the assumption that f is absolutely integrable, and since $e^{v\cdot x}$ is rapidly decreasing, so is the product $(-ix)^N e^{v\cdot x}$. It now follows that $\hat{f}$ has derivatives of all orders in the lower half of the complex plane, where the N^{th} derivative is given as the Fourier transform of $(-ix)^N f(x)$. ∎

More generally, we can see from the proof that if f is measurable and the integral in (31) converges for some $v = v_0$, then it converges in the closed half plane $\Im(\xi) \leq v_0$, and $\hat{f}(\xi)$ is analytic in the open halfspace $\{\xi \in \mathbf{C} : \Im(\xi) < v_0\}$ (Widder, 1971, or Papoulis, 1977). If f vanishes at infinity, then the more rapidly it decays, the larger the open halfspace becomes in which $\hat{f}$ is analytic. Conversely, if f grows without bound at infinity, (so $v_0 < 0$) then the more rapidly f increases, the smaller the open half plane of analyticity of $\hat{f}$.

Theorem 0.10 *Let f be a measurable function such that*

$$f(x)e^{Ax} \in \mathcal{L}^1 \tag{33}$$

for some fixed constant $A \in \mathbf{R}$. Then for each fixed $\nu \leq A$, the function f can be recovered as

$$f(x) = \frac{1}{2\pi}\int_{L_\nu} \hat{f}(\xi)e^{i\xi}\,d\xi, \tag{34}$$

where L_ν is the horizontal line passing through $i\nu$ in the complex plane.

Proof Fix any $\nu \leq A$. Then $g_\nu(x) \equiv f(x)e^{\nu x} \in \mathcal{L}^1$, and its Fourier transform is given as

$$\hat{g}_\nu(\omega) = \int_{\mathbf{R}} f(x)e^{-ix(\omega+i\nu)}dx \equiv \hat{f}(\omega + i\nu). \tag{35}$$

Applying the inverse Fourier transform yields

$$f(x)e^{\nu x} = g_\nu(x) = \frac{1}{2\pi}\int_{\mathbf{R}} \hat{f}(\omega + i\nu)e^{i\omega x}d\omega, \quad \text{or} \tag{36}$$

$$f(x)=\frac{1}{2\pi}\int_{\mathbf{R}}\hat{f}(\omega+i\nu)e^{ix(\omega+i\nu)}d\omega=\frac{1}{2\pi}\int_{L_\nu}\hat{f}(\xi)e^{i\xi}\,d\xi, \tag{37}$$

where $\xi=\omega+i\nu,\ d\xi=d\omega$, and L_ν is the horizontal line passing through $i\nu$ in the complex plane. ∎

This result is a Fourier-domain analogue to Bromwich's* method for recovering the inverse Laplace transform of $F(s)$ by integrating F along any vertical line, or Bromwich path, (Papoulis, 1977) to the right of all singularities of F.

Minimum phase

Definition 0.11 *A tempered distribution (or function) $s\in\mathcal{S}'(\mathbf{R})$ is minimum phase if $\hat{s}$ and $1/\hat{s}$ correspond to analytic functions on the open lower half of the complex plane,*

$$\mathbf{C}_-=\{u+iv\in\mathbf{C}:v<0\}. \tag{38}$$

Actually, the term "minimum phase" more appropriately relates to the Fourier spectrum $\hat{s}$, whereas s is more accurately described as being "minimum delay". The two terms have come to have the same meaning in the Geophysics community (e.g., Sheriff, 1991), reasonably so due to the identification of $\mathcal{L}^2(\mathbf{R})$ with itself effected by the Fourier transform. We shall continue with this convention. We choose to forego a discussion of the physical implications of the definition, as it is beyond the scope of our present objectives. The interested reader should consult the appendices in Robinson's book (1967), as well as the primary author's Ph. D. thesis (to appear in Spring, 2005).

According to our definition, $\hat{s}$ is minimum phase, if and only if, $1/\hat{s}$ is minimum phase. This is the main reason for demanding such generality of s. Indeed, if s is a signal, then its Fourier transform vanishes at infinity: thus $1/\hat{s}$ is unbounded, and its Fourier transform, $\mathcal{F}^{-1}\{1/\hat{s}\}$, provided it exists, cannot be a function. Quite naturally, we want to preserve the symmetry implied by the definition, namely, that s should be minimum phase if and only if $\mathcal{F}^{-1}\{1/\hat{s}\}$ is minimum phase.

Example The function $f(x)=h(x)e^{-ax}$, for each $a>0$, is a minimum phase signal. Indeed, $f\in\mathcal{L}^2(\mathbf{R})\cap\mathcal{L}^1(\mathbf{R})$, so f is a signal and its Fourier transform can be computed using formula (10) as

$$\hat{f}(\xi)=\int_{\mathbf{R}}h(x)e^{-x(a+i\xi)}dx=\int_0^\infty e^{-x(a+i\xi)}dx=\frac{1}{a+i\xi}, \tag{39}$$

provided $a>0$. Certainly, $\hat{f}$ and $1/\hat{f}$ are tempered functions on $\mathbf{R}$, and so they correspond to tempered distributions defined as in (14). Now $\hat{f}$ is analytic everywhere except at the point $\xi=ai$ in the upper half plane, and $1/\hat{f}$ is entire, which proves the claim. ∎

*Thomas John l'Anson Bromwich, 1875-1929

Example It is instructive to go through the case $a = 0$, namely, to show that the Heaviside function is minimum phase. As in (21), we have

$$\hat{h}(\omega) = \pi\delta(\omega) - \frac{i}{\omega}. \tag{40}$$

Now δ is supported at the origin, and so on the open lower half of the complex plane, $\hat{h}$ agrees with the function

$$\hat{h}'(\xi) = -\frac{i}{\xi}, \quad \Im(\xi) < 0. \tag{41}$$

This function and its inverse are analytic for $\Im(\xi) < 0$, and the claim follows. ∎

The previous example illustrates why, in our definition of minimum phase, we do not require $\hat{s}$ or its inverse to be analytic on the real line.

Example The Dirac distribution is minimum phase. Its Fourier transform is the distribution corresponding to the constant function, 1, whose extension to the whole complex plane is entire. ∎

Example More generally, for any complex numbers μ and $\epsilon \neq 0$, the tempered distribution $\mu + \epsilon\delta$ is minimum phase. Indeed, its Fourier transform is $2\pi\mu\delta + \epsilon$, which agrees with the constant function ϵ everywhere except at the origin of $\mathbf{C}$. The condition $\epsilon \neq 0$ is necessary to ensure $1/\epsilon$ is analytic. ∎

Example Suppose g is a causal Schwartz function, and consider the function of frequency obtained by exponentiating its Fourier transform (recall that $\hat{g}$ is also a Schwartz function):

$$\exp\left(\hat{g}(\omega)\right) = \exp\left(\Re(\hat{g}(\omega))\right) \exp\left(i\,\Im(\hat{g}(\omega))\right). \tag{42}$$

Since g is causal and certainly in $\mathcal{L}^1(\mathbf{R})$, Proposition 0.9 asserts that $\pm\hat{g}$ are analytic functions on the open lower half plane; hence so are $\exp\left(\hat{g}\right)$ and $1/\exp\left(\hat{g}\right) = \exp\left(-\hat{g}\right)$, since the exponential function is entire. In particular, the tempered (actually bounded) function $\exp\left(\hat{g}\right)$ is the Fourier spectrum of a minimum-phase tempered distribution $T \in \mathcal{S}'(\mathbf{R})$.

By Proposition 0.7, since $g \in \mathcal{S}(\mathbf{R}) \subset \mathcal{L}^1(\mathbf{R})$ is causal, the real and imaginary parts of its Fourier transform are related by the Hilbert transform:

$$\Re(\hat{g}) = \mathcal{H}\Im(\hat{g}), \text{ or } \Im(\hat{g}) = -\mathcal{H}\,\Re(\hat{g}). \tag{43}$$

In particular, we see that the phase and magnitude spectra of T are related by

$$\phi_T = \Im(\hat{g}) = \mathcal{H}\ln(|\hat{T}|). \quad ∎ \tag{44}$$

Example The zero function is not minimum phase. Indeed, its Fourier transform is again zero, which is entire, but nowhere invertible. ∎

We should expect the property of minimum phase—for a good definition of minimum phase—to be translation invariant, that is, independent of the choice of origin for the time axis.

Proposition 0.12 *A signal f is minimum phase if and only if each of its translates $T_\tau f$, $(\tau \in \mathbf{R})$ is minimum phase.*

Proof Suppose f is minimum phase, so $\hat{f}$ and $1/\hat{f}$ are analytic on the open lower half of the complex plane. Since $(\widehat{T_\tau f})(\omega) = \exp(i\omega\tau)\hat{f}(\omega)$, and $\exp(i\omega\tau)$ and its inverse $\exp(-i\omega\tau)$ are entire functions (that is, analytic everywhere on the complex plane), it follows that $\widehat{T_\tau f}$ and $1/\widehat{T_\tau f}$ are analytic on the open lower half plane, whence $T_\tau f$ is minimum phase for all $\tau \in \mathbf{R}$. The converse follows obviously by setting $\tau = 0$. ∎

The following result offers an alternate characterization of minimum-phase tempered distributions.

Proposition 0.13 *A tempered distribution s is minimum phase if and only if $\ln \hat{s}$ is analytic on the open lower half of the complex plane.*

Proof The complex logarithm is an analytic function everywhere except at the origin. Thus $\ln \hat{s}$ is analytic if and only if $\hat{s}$ is analytic and non-zero. Also, $1/\hat{s}$ is analytic if and only if $\hat{s}$ is analytic and non-zero. Thus s is minimum phase if and only if $\ln \hat{s}$ is an analytic function on the open lower half of the complex plane. ∎

The following results rely on Jordan's Lemma, a result from complex analysis, which we have included in the Appendix as Proposition .22.

Theorem 0.14 *If $F(\omega)$ is analytic for $\Im(\omega) \leq 0$ and satisfies the hypotheses of Proposition .22, then $F(\omega)$ is the Fourier transform of a causal tempered distribution $f \in \mathcal{S}'(\mathbf{R})$.*

Proof First, F is bounded on $\mathbf{R}$, since $F(\omega) \to 0$ as $R \to \infty$. Thus F is tempered, so that $F \in \mathcal{S}'(\mathbf{R})$, and we have that $f \in \mathcal{S}'(\mathbf{R})$, where f is the inverse Fourier transform of F. For each $R > 0$, let $\gamma = \gamma_R$ be the semicircle of radius R in the closed lower half of $\mathbf{C}$, centred at the origin, and let $\Gamma = \Gamma_R$ be the simple closed curve connecting γ with the interval $[-R, R]$. Since $F(\omega)$ is analytic for $\Im(\omega) \leq 0$, and $e^{i\omega t}$ is entire, Cauchy's Theorem .20 asserts that

$$\int_\Gamma F(\omega)e^{i\omega t} d\omega = 0. \tag{45}$$

By Proposition .22, for each $t < 0$ the integrals $\int_\gamma F(\omega)e^{i\omega t} d\omega$ converge to 0 as $R \to \infty$. In particular, for each $t < 0$, the integrals defined by

$$\int_{-R}^{R} F(\omega)e^{i\omega t} d\omega = \int_\Gamma F(\omega)e^{i\omega t} d\omega - \int_\gamma F(\omega)e^{i\omega t} d\omega = -\int_\gamma F(\omega)e^{i\omega t} d\omega \tag{46}$$

form a bounded net, indexed by $R > 0$, which converges to 0 as $R \to \infty$. Thus, for all $t < 0$, passing to the limit as $R \to \infty$ shows that

$$\int_{\mathbf{R}} F(\omega) e^{i\omega t} d\omega = 0, \tag{47}$$

and it follows that the distribution f agrees with the zero function on $(-\infty, 0)$, that is, f is causal. ∎

Corollary 0.15 *If $f \in \mathcal{S}'(\mathbf{R})$ is minimum phase and its Fourier transform, $F \in \mathcal{S}'(\mathbf{R})$, satisfies the hypotheses of Proposition .22, then $\mathcal{H}F = iF$.*

Proof Since f is minimum phase, it is analytic in the lower half plane. By hypothesis, it follows from Theorem 0.14 that f is causal. By definition of the Hilbert transform of a causal distribution, $\mathcal{H}F = iF$. ∎

Theorem 0.16 *If $f \in \mathcal{S}'(\mathbf{R})$ is minimum phase, and $\ln \hat{f}$ satisfies the hypotheses of Proposition .22, then $\mathcal{H}(\ln \hat{f}) = i \ln \hat{f}$.*

Proof By Theorem 0.14, $ln \hat{f}$ corresponds to a causal tempered distribution.

Theorem 0.17 *(extension of Signal Front result in Papoulis, 1977) If $\hat{f} \in \mathcal{S}'(\mathbf{R})$ is analytic in the open lower half plane, and if for some $\tau \in \mathbf{R}$,*

$$e^{i\tau\xi} \hat{f}(\xi) \to 0 \;\; as \;\; \xi \to \infty \;\; on \;\; \gamma_R, \tag{48}$$

(as in Proposition .22) then Supp $(f) \subseteq [\tau, \infty)$.

Proof Consider the translate, $T_{-\tau} f$ of f by $-\tau$. Then $\{T_{-\tau} f\}^\wedge (\xi) = e^{i\tau\xi} \hat{f}(\xi)$ is analytic in the open lower half plane and satisfies the hypotheses of Proposition .22. Thus, by Theorem 0.14, Supp $(f) \subseteq [0, \infty)$, and the result follows.

Definition 0.18 *(extension of Signal Front concept (Papoulis, 1977) to tempered distributions) Given $f \in \mathcal{S}'(\mathbf{R})$ satisfying (48) for some $\tau \in \mathbf{R}$, the supremum, τ_0, over the set of all τ satisfying (48) is called the signal front of f.*

Of course, f need not be a signal in the classical sense. Also, $\tau_0 = \infty$ if and only if $f \equiv 0$.

MINIMUM-PHASE FILTERS

Definition 0.19 *A linear operator L is minimum phase if it preserves minimum phase: that is, if, for each minimum phase tempered distribution, $s \in \mathcal{S}'(\mathbf{R})$ for which Ls defines a tempered distribution, Ls is again minimum phase.*

The domain of the operator L is intentionally left unspecified, since it depends on the regularity of the Schwartz kernel of L. We thus consider L as a mapping from the preimage, $L^{-1}\{\mathcal{S}'(\mathbf{R})\}$, of $\mathcal{S}'(\mathbf{R})$ into $\mathcal{S}'(\mathbf{R})$.

Example Given any minimum phase Schwartz class function $f \in \mathcal{S}(\mathbf{R})$, consider the convolution operator $L_f : \mathcal{S}'(\mathbf{R}) \to \mathcal{S}'(\mathbf{R})$ defined as $L_f s = f * s$. The product $\hat{f}(\xi)\hat{s}(\xi)$ and its pointwise inverse are analytic functions in the open lower half plane, since this is true of both $\hat{f}$ and $\hat{s}$ by assumption. In particular, $L_f s \in \mathcal{S}'(\mathbf{R})$ is minimum phase, and so L_f is a minimum phse linear operator.

If the kernel f were in $\mathcal{L}^1(\mathbf{R})$, then we would have a bounded (continuous) minimum-phase linear operator on $\mathcal{L}^2(\mathbf{R})$. Since $\mathcal{L}^2(\mathbf{R}) \subset \mathcal{S}'(\mathbf{R})$, L_f can be viewed—in a trivial way—as an operator $L_f : \mathcal{L}^2(\mathbf{R}) \to \mathcal{S}'(\mathbf{R})$. ∎

The convolution operators are commonly referred to as *filters* in signal processing and imaging. They represent time-invariant, or stationary, linear systems.

Of much greater interest to us in our research is *nonstationary* filtering. Indeed many, if not all, linear physical systems exhibit nonstationarity and thus cannot properly be described by convolution operators.

To this end, consider a collection $f = \{f_\tau : \tau \in \mathbf{R}\}$ of minimum-phase tempered distributions, parameterized by $\tau \in \mathbf{R}$. This just describes a function $f = f(\tau, t)$ of two variables whenever each member is itself a function. Alternatively, we can interpret the Fourier transform of f as a family of filter kernels, $\hat{f} = \{\hat{f}_\tau : \tau \in \mathbf{R}\}$, also parameterized by $\tau \in \mathbf{R}$. Specifically, the action of the resulting family of filters can be represented (at least formally) by the following inverse Fourier transform (see Kohn and Nirenberg, 1965, and Stein, 1993):

$$(L_f s)(\tau) = \frac{1}{2\pi}\int_{\mathbf{R}} \hat{f}(\tau,\omega)\hat{s}(\omega)e^{i\tau\omega}d\omega, \quad \text{where } s \in L_f^{-1}\{\mathcal{S}'(\mathbf{R})\}. \tag{49}$$

For each fixed τ, and for each minimum phase $s \in L_f^{-1}\{\mathcal{S}'(\mathbf{R})\}$, the product $\hat{f}_\tau \hat{s}$ and its pointwise inverse are analytic functions in the open lower half of the complex plane, and thus represent (via an inverse Fourier transform) a minimum-phase tempered distribution; namely, $L_f s$. Thus L_f defines a nonstationary, minimum-phase linear operator.

CONCLUSIONS

We provide an introduction to the various function spaces used for representing signals and images, as well as a discussion on the space of tempered distributions. This fixed the notation and concepts required for the rest of the paper.

Our main objective was to extend the concept of minimum phase to the analog setting. To this end, we proposed a logically consistent and mathematically rigorous theory in the framework of tempered distributions.

This theory allowed us to extend the concept of a minimum-phase filter (linear operator) to the analog setting, including nonstationary analog systems.

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APPENDIX: RESULTS FROM COMPLEX ANALYSIS

Theorem .20 *(Cauchy-Goursat Theorem, e.g., Churchill and Brown, 1984) If the function F is analytic at all points interior to and on a simple closed curve Γ, then*

$$\int_{\Gamma} F(z)\, dz = 0. \tag{50}$$

Lemma .21 *(Jordan's Inequality, adapted from Churchill and Brown, 1984) For each $R > 0$,*

$$\int_{-\pi/2}^{0} e^{R\sin\theta}\, d\theta < \frac{\pi}{2R}. \tag{51}$$

Proof (Churchill and Brown, 1984) For $-\pi/2 \le \theta \le 0$, we have $0 \ge 2\theta/\pi \ge \sin\theta$. Thus, for any $R > 0$ and $\theta \in [-\pi/2, 0]$, $e^{R\sin\theta} \le e^{2R\theta/\pi}$, so that

$$\int_{-\pi/2}^{0} e^{R\sin\theta}\, d\theta \le \int_{-\pi/2}^{0} e^{2R\theta/\pi}\, d\theta = \frac{\pi}{2R}(1 - e^{R}) < \frac{\pi}{2R}. \quad \blacksquare \tag{52}$$

Proposition .22 *(Jordan's Lemma, adapted from (Papoulis, 1977)) Let $\gamma = \gamma_R$ be the semicircle of radius R in the lower half of $\mathbf{C}$, centered at the origin. If $F(\omega)$ is a measurable function such that for all $\omega \in \gamma$, $F(\omega) \to 0$ as $R \to \infty$, then for all $t < 0$,*

$$\int_{\gamma} F(\omega) e^{i\omega t} d\omega \to 0 \;\; \textit{as } R \to \infty. \tag{53}$$

Proof Let $\epsilon > 0$ be given. Since $F(\omega) \to 0$ as $R \to \infty$, (for $\Im(\omega) \le 0$) we can choose $R_0 > 0$ such that $|F(\omega)| < \epsilon$ for $|\omega| > R_0$. With the change of variables $\omega = Re^{i\theta}$, $d\omega = iRe^{i\theta} d\theta$, we have that for all $R > R_0$ and $t < 0$,

$$\begin{aligned} \left| \int_{\Gamma} F(\omega) e^{i\omega t} d\omega \right| &\le R \int_{-\pi}^{0} \left| e^{iRt(\cos\theta + i\sin\theta)} f(Re^{i\theta}) \right| d\theta \\ &\le \epsilon R \int_{-\pi}^{0} \left| e^{-Rt(\sin\theta)} \right| < \frac{\epsilon\pi}{-t}, \end{aligned}$$

where the last inequality follows from Lemma .21, together with the fact that $\sin\theta$ is symmetric about the vertical line through $-\pi/2$; and hence that the integral over $[-\pi, 0]$ is twice that over $[-\pi/2, 0]$. ∎

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