

Symbols “R” Us: Seismic Imaging, One-Way Wave Equations, Pseudodifferential Operators, Path Integrals, and all that Jazz

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ABSTRACT

In this report we summarize an extension of Fourier analysis for the solution of the wave equation with a non-constant coefficient corresponding to an inhomogeneous medium. The underlying physics of the problem is exploited to link pseudodifferential operators and phase space path integrals to obtain a marching algorithm that incorporates the backward scattering into the evolution of the wave. This allows us to successfully apply single-sweep, one-way marching methods in inherently two-way environments, which was not achieved before through other methods for this problem.

INTRODUCTION

Understanding wavefield propagation and the construction of efficient, approximate numerical algorithms have been important issues in many industrial settings for many years. Mathematicians, physicists, and other applied scientists, in fields as diverse as ocean acoustics, electromagnetics, medical imaging, optical design, and seismic exploration, have struggled with these issues. Wave propagation in complex environments is a classical problem that has been attacked by a variety of approaches over the years, including (1) direct wavefield approximations (e.g., perturbation theory, asymptotic ray theory, Gaussian beams) and (2) computational partial differential equation (pde) methods (e.g., finite differences, finite elements, spectral methods, wavelets). However, these approaches can often be problematic. Approximation methods often offer some physical insight, but they may be limited to specific parameter regimes, not properly accounting for the underlying physics outside of these regimes. Computational pde methods, in the frequency domain, in particular, can often be prohibitively expensive and time consuming for large scale problems involving large data sets. Moreover, the computational pde methods do not, in general, reveal the underlying physics in a transparent manner.

Rather than considering a wide range of wave propagation formulations in this project, we will focus on seismic imaging applications. The problem is illustrated in Figure 1, which shows the Marmousi synthetic velocity model for an area off the coast of West Africa. Even though this is a relatively simple, two-dimensional, constant density, scalar model, there is complex fine layering, fault lines, velocities ranging from 1500 *m/s* to

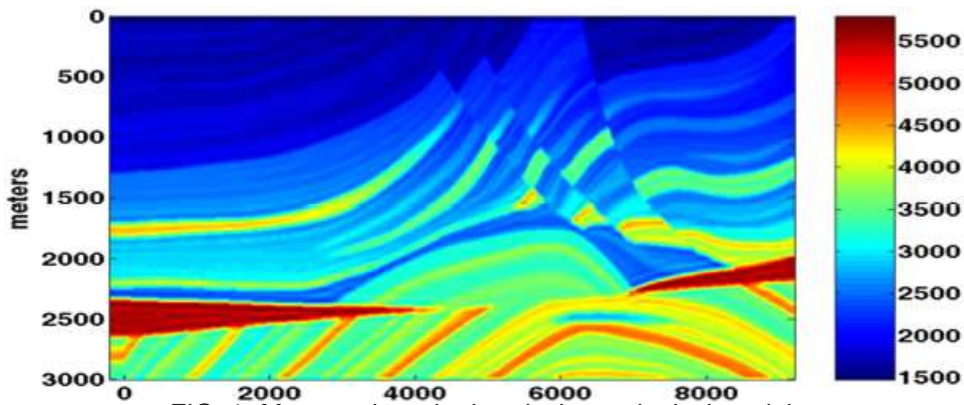


FIG. 1. Marmousi synthetic velocity geological model.

almost 6000 m/s , regions with very strong gradients, a reservoir, and regions exhibiting pronounced focusing and defocusing phenomena. The idea is to set off explosive sources along the surface, and from the collected reflection data along the surface, construct an appropriate image of the substructure. In other words, we want to construct images of the Marmousi model, from the synthetic data, that accurately reflect the complex model structure (at least to the extent that this inverse process will allow). In the seismic industry, of course, these reconstructions are done with actual field data sets, which inherently contain limited and noisy data. An image of this model computed in 1998 is illustrated in Figure 2.

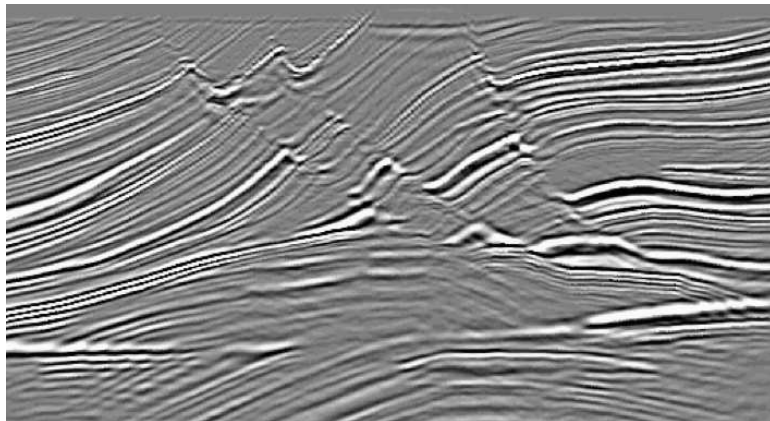


FIG. 2. Image of Marmousi model computed in 1998. Figure from Romero et al. (2000)

One of the most commonly used techniques for seismic imaging is seismic depth migration. Kirchhoff migration is based on applying asymptotic ray theory for the wave propagation process. While the workhorse of the seismic industry, and very appealing from a physics standpoint, this high-frequency asymptotic method is inadequate for modelling wave propagation in Marmousi-type environments at the lower end of typical experimental seismic bandwidths (say between 10 – 30 Hz). This can be contrasted with wave equation migration methods based on computational pde methods and the locally-homogeneous-medium (LHM) approximation to the wavefield extrapolator (the Generalized Phase Shift Plus Interpolation (GPSPI) algorithm). Computational pde methods can often result in excessively large linear systems. The GPSPI algorithm is based on the industry belief that the LHM approximation to the wavefield extrapolator is an exact statement of the global propagation process (at least for range-independent models). This, unfortunately, is not the case outside of a strictly homogeneous medium.

The specific problem considered is the following. Construct a wave propagation model, at the level of the fixed-frequency, scalar Helmholtz equation, which can form the basis for wave equation migration algorithms. This wave propagation model should be a full-wave model, so as to be appropriate over the entire seismic bandwidth, and computationally faster and physically more transparent than pde-based methods, all the while providing a complete mathematical framework in which to examine, correct, and extend the GPSPI algorithm. It would also be very convenient if the computational realization of this model ran (in principle) exactly in the same manner as the GPSPI algorithm. In other words, we want to combine the physical insight and relative computational speed of GPSPI with the numerical accuracy of a complete pde-solver-based algorithm. Essentially, we want the best of both worlds for a seismic migration algorithm.

We will approach this problem through the application of what is loosely referred to as “phase space and path integral” methods, which were developed primarily in the quantum physics and theoretical mathematics (pde) communities. The principal aims of this approach are to (1) exploit well-posed, one-way marching methods in inherently two-way global problems, (2) exploit the correspondences between classical wave propagation, quantum physics, and microlocal analysis (modern mathematical asymptotics), and (3) extend Fourier analysis to inhomogeneous environments. Since this is a brief report, and these “phase space and path integral” methods have a rather involved and extensive literature, we will introduce them essentially through the homogeneous medium case, and rely on the citations in the References section for the detailed story.

HOMOGENEOUS MEDIUM

We begin with the full wave equation which models the propagation of waves in a medium. For simplicity we will work in two dimensions (everything can be extended to arbitrary dimension).

$$\left(\partial_x^2 + \partial_z^2 - \frac{1}{c^2(\vec{x})} \partial_t^2 \right) \psi(\vec{x}, t) = S(\vec{x} - \vec{x}_s, t - t_0), \quad (1)$$

where x is the propagation direction, z is the transverse direction, $c(\vec{x})$ is the speed of sound in the medium, $S(\vec{x}, t)$ is a source term, t is time, and $\vec{x} = (x, z)$.

We consider a wavefield $\psi(\vec{x}, t) = \phi(\vec{x})e^{-i\omega t}$, where ω is the frequency, and assume that the source $S(\vec{x}, t)$ is a delta distribution in both position and time. By taking a Fourier transform in the time variable we obtain the Helmholtz equation, which describes the propagation of a single frequency wave:

$$(\partial_x^2 + \partial_z^2 + \bar{k}^2 K^2(\vec{x})) \phi(\vec{x}) = -\delta(\vec{x} - \vec{x}_s), \quad (2)$$

where $K(\vec{x})$ is the refractive index of the medium at the point \vec{x} , and $\bar{k} = \omega/c_0$.

To gain physical insight, we consider a homogeneous medium in which $K(\vec{x}) = K_0$, and solve (2) away from the source:

$$(\partial_x^2 + \partial_z^2 + \bar{k}^2 K_0^2) \phi(x, z) = 0. \quad (3)$$

We can formally separate the solution into two non-interacting waves: the right-going wave ϕ^+ , and the left-going wave ϕ^- :

$$\left(\frac{i}{\bar{k}} \partial_x + \mathbf{B} \right) \phi^+(x, z) = 0, \quad (4)$$

and

$$\left(-\frac{i}{\bar{k}} \partial_x + \mathbf{B} \right) \phi^-(x, z) = 0, \quad (5)$$

where

$$\mathbf{B} \equiv \left(K_0^2 + \frac{1}{\bar{k}^2} \partial_z^2 \right)^{1/2}. \quad (6)$$

The equations above are a formal factorization of expression (3). To make sense of \mathbf{B} , consider the Fourier transform of (3). The equation for $\phi^+(x, z)$ is as follows:

$$\frac{i}{\bar{k}} \partial_x \phi^+(x, z) + \int_R dp (K_0^2 - p^2)^{1/2} e^{i\bar{k}pz} \hat{\phi}^+(x, p) = 0. \quad (7)$$

where $\hat{\phi}^+(x, p)$ is the Fourier transform of $\phi^+(x, z)$. Similarly, we can write an equation for $\phi^-(x, z)$.

It is well known that the fundamental solution to this one way wave equation can be represented essentially as a first order Hankel function of the first kind, which can be expressed by the following identity:

$$G^+(x, z|0, z') = \lim_{N \rightarrow \infty} \int_{R^{2N-1}} \prod_{j=1}^{N-1} dz_j \prod_{j=1}^N \frac{\bar{k}}{2\pi} dp_j \cdot \exp \left[i\bar{k} \sum_{j=1}^N \left[p_j(z_j - z_{j-1}) + \frac{x}{N} (K_0^2 - p_j^2)^{1/2} \right] \right]. \quad (8)$$

The fundamental solution representation in (8) may appear rather involved for what is essentially a Hankel function identity. However, it is exactly in the form of a phase space path integral representation from quantum physics, and it is this form that will be used to represent the wavefield solutions in the more general inhomogeneous cases. Moreover, the form of (8) immediately results in a marching computational algorithm.

We can use the above expression to construct a marching algorithm, as follows:

$$\phi^+(x + \Delta x, z) = \int_R dp e^{i\bar{k}pz} \left[e^{i\bar{k}\Delta x (K_0^2 - p^2)^{1/2}} \hat{\phi}^+(x, p) \right]. \quad (9)$$

We recognize this as a formula involving forward and inverse Fourier transforms:

$$\phi^+(x + \Delta x, z) = F^{-1} \left[e^{i\bar{k}\Delta x (K_0^2 - p^2)^{1/2}} F [\phi^+(x, z), p], z \right]. \quad (10)$$

This algorithm is simple, well-posed, and incorporates all the physics of the homogeneous problem.

TRANSVERSELY-INHOMOGENEOUS MEDIUM

Now we consider a slightly more complex case than the completely homogeneous medium. We introduce dependence of K along the z axis, the transverse direction. Solving this problem will naturally lead us to the solution of the general problem in which we also have dependence on x , the propagation direction.

As in the homogeneous case, right- and left-going waves are decoupled. In both the transversely-inhomogeneous and homogeneous medium cases, there are physical, right- and left-travelling wave fields, as follows from simple separation of variables arguments. An explicit representation of (4), written for the transversely-inhomogeneous case, is provided by the theory of pseudodifferential operators from the modern mathematical theory of asymptotics, in addition to more formal operator construction methods developed in the quantum physics community. These developments enable the explicit construction of non-trivial functions of non-commuting operators, such as the formal square-root operator in (6) with the constant index of refraction replaced by its local, transversely-inhomogeneous value. The equation for the right-going wave is given in the Weyl pseudodifferential operator calculus by:

$$\begin{aligned} & \frac{i}{\bar{k}} \partial_x \phi^+(x, z) + \\ & \frac{\bar{k}}{2\pi} \int_{R^2} dp dz' \Omega_{\mathbf{B}} \left(p, \frac{z + z'}{2} \right) \exp(i\bar{k}p(z - z')) \phi^+(x, z') = 0, \end{aligned} \quad (11)$$

where the operator symbol $\Omega_{\mathbf{B}}$ satisfies the composition equation:

$$\begin{aligned} \Omega_{\mathbf{B}^2}(p, q) &= K^2(q) - p^2 \\ &= \left(\frac{\bar{k}}{\pi} \right)^2 \int_{R^4} dt ds dy dw \\ &\quad \cdot \Omega_{\mathbf{B}}(t + p, s + q) \Omega_{\mathbf{B}}(y + p, w + q) \exp(2i\bar{k}(sy - tw)). \end{aligned} \quad (12)$$

The form of (11) is exactly the same as in the homogeneous case, except that the square root function is replaced by $\Omega_{\mathbf{B}}$. In the homogeneous medium limit, $\Omega_{\mathbf{B}}$ reduces to $(K_0^2 - p^2)^{1/2}$. This is a representation of the formal square root operator from the Weyl pseudodifferential calculus. The operator is defined through its square, as is the case in functional analysis.

It may seem strange that we are effectively replacing a linear partial differential equation with a quadratically nonlinear, nonlocal composition equation (12). However, it will turn out that these symbols $\Omega_{\mathbf{B}}$ can actually image the environments directly. Moreover, approximations made at the level of the symbol will have a far greater range of validity than the corresponding approximations made at the level of the wavefield. These points are discussed in the literature citations.

We have the same path integral and algorithm as the one we described for the homogeneous case. The difference is that $(K_0^2 - p^2)^{1/2}$ is replaced by:

$$h_{\mathbf{B}}^s(p, q) = \left(\frac{\bar{k}}{\pi}\right) \int_{R^2} ds du \Omega_{\mathbf{B}}(s, u) e^{-2i\bar{k}(q-u)(p-s)}. \quad (13)$$

While exact solutions for specific profiles $K^2(z)$ have been constructed, applications to seismo-acoustic wave propagation, imaging, and inversion will depend upon uniform asymptotic expansions of the operator symbol. For $\Omega_{\mathbf{B}}(p, q)$ this takes the general form:

$$\Omega_{\mathbf{B}}(p, q) \sim (K^2(q) - p^2)^{1/2} + \text{uniform terms through } O(1/\bar{k}^2), \quad (14)$$

in the high-frequency ($\bar{k} \rightarrow \infty$) limit. The details of the exact solution constructions and the explicit form and derivation of (14) are best left to the literature citations. The most important point is the following. While the modern mathematical theory of asymptotics and constructions from quantum physics provide the necessary mathematical framework to explicitly derive our equations, the asymptotic solution of the composition equation (12) lies outside of both of these areas. The reason for this is, essentially, that these theories and examples from quantum physics are based on propagation of singularities arguments (think time domain wave propagation), while the Helmholtz equation is a smoothing problem (think frequency domain wave propagation). Constructions like (14) require going beyond the modern mathematical theory of asymptotics and examples from quantum physics.

Assuming that we can compute $\Omega_{\mathbf{B}}$ numerically, or asymptotically approximate it, then we can use it in the following marching algorithm:

$$\phi^+(x + \Delta x, z) \approx \int_R dp \exp(i\bar{k}pz) \left[\exp(i\bar{k}\Delta x h_{\mathbf{B}}^s(p, z)) \hat{\phi}^+(x, p) \right], \quad (15)$$

or, in Fourier transform notation:

$$\phi^+(x + \Delta x, z) \approx F^{-1} \left[\exp(i\bar{k}\Delta x h_{\mathbf{B}}^s(p, z)) F [\phi^+(x, z), p], z \right]. \quad (16)$$

GENERALLY-INHOMOGENEOUS MEDIUM

In the homogeneous and transversely-inhomogeneous media we were able decouple the right- and left-going waves. We were able to formally factor the Helmholtz equation.

However, in a generally-inhomogeneous medium, in which K depends on both the transverse and propagation directions, the right- and left-going waves are inherently coupled. In a general inhomogeneous medium, backscattering may be present – the problem becomes global in nature (see Figure 3). The problem is inherently two-way. It is straightforward to construct one-way wave equations in the transversely-inhomogeneous and homogeneous cases since the physics already gives us independent one-way wave equations. But how can we construct one-way wave equations in this general case? The answer lies in considering the general scattering problem for the geometry of Figure 3. Applying invariant imbedding methods enables the construction of the reflection and transmission operators associated with the generally-inhomogeneous block. Transforming this scattering picture into a boundary-value picture, in terms of Dirichlet-to-Neumann operators and the total wavefield and its normal derivative, produces the desired result. We get a one-way equation to construct the operator, and a one-way wave equation, in terms of the operator, governing the total wavefield. The detailed treatment can be found in the literature citations.

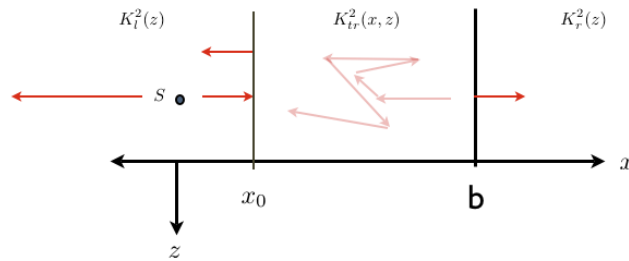


FIG. 3. Schematic of inhomogeneous block.

In a similar manner to the homogeneous and transversely-inhomogeneous models, the right-going field is governed by:

$$\left(\frac{1}{\bar{k}} \partial_x + \Lambda^+(x, b) \right) \phi(x, z) = 0, \quad (17)$$

(and similarly for the left-propagating field). From (17) we note that Λ^+ is the Dirichlet-to-Neumann operator. In fact the square root operator \mathbf{B} is the transversely-inhomogeneous DtN operator as well.

The one-way wave equation is:

$$\frac{1}{\bar{k}} \partial_x \phi(x, z) + \frac{\bar{k}}{2\pi} \int_{\mathbb{R}^2} dp dz' \Omega_{\Lambda^+} \left(x, b; p, \frac{z+z'}{2} \right) \exp(i\bar{k}p(z-z')) \phi(x, z') = 0 \quad (18)$$

where Ω_{Λ^+} is the symbol in the Weyl calculus corresponding to Λ^+ .

The $\Lambda^+(x, b)$ operator encapsulates the two-way scattering behaviour of the block between x and b . That is, $\Lambda^+(x, b)$ contains all information about the block's internal reflections and transmissions between x and b . By solving the series of internal scattering

problems that the wavefield encounters as it propagates into the block, the inherently coupled problem is decoupled into independent wavefield components, resulting in the desired one-way wave equation. We march one way (from b to x) to construct Λ^+ (which describes the two-way wave behaviour), and then we march the other way to obtain the total wavefield.

A short, formal derivation of the one-way equation defining the DtN operator is the following. Taking the partial derivative with respect to x of equation (17), applying (17), and using the original Helmholtz equation result in the marching equation for Λ^+ is:

$$\frac{1}{\bar{k}} \partial_x \Lambda^+(x, b) = (\Lambda^+(x, b))^2 + \mathbf{B}^2(x), \quad (19)$$

with the initial condition: $\Lambda^+(b, b) = -i\mathbf{B}(b)$.

The composition equation for Ω_{Λ^+} is:

$$\begin{aligned} \frac{1}{\bar{k}} \partial_x \Omega_{\Lambda^+}(x, b; p, q) = & \\ & \left(\frac{\bar{k}}{\pi}\right)^2 \int_{\mathbb{R}^4} dt ds dy dw \Omega_{\Lambda^+}(x, b; t + p, s + q) \\ & \cdot \Omega_{\Lambda^+}(x, b; y + p, w + q) \exp(2i\bar{k}(sy - tw)) + K^2(x, q) - p^2, \end{aligned} \quad (20)$$

with initial condition: $\Omega_{\Lambda^+}(b, b; p, q) = -i\Omega_{\mathbf{B}}(b; p, q)$. Equation (20) is just the expression of (19) in the Weyl pseudodifferential operator calculus.

As in the transversely-inhomogeneous case, we avoid solving the composition equation (20) by applying uniform asymptotics. While computational algorithms exist that solve nonlocal Riccati-type equations like (20), the uniform asymptotic approximation of the operator symbol is the key to a computationally feasible algorithm. More details can be found in the literature citations.

We have extended the homogeneous Fourier analysis to the general inhomogeneous case. The path integral representation of the fundamental solution is:

$$\begin{aligned} G^+(x, z|x_0, z') = & \lim_{N \rightarrow \infty} \int_{\mathbb{R}^{2N-1}} \prod_{j=1}^{N-1} dz_j \prod_{j=1}^N \frac{\bar{k}}{2\pi} dp_j \\ & \cdot \exp \left[i\bar{k} \sum_{j=1}^N \left[p_j(z_j - z_{j-1}) + i \frac{\Delta x}{N} h_{\Lambda^+}^s(x_j, b; p_j, z_j) \right] \right], \end{aligned} \quad (21)$$

where

$$h_{\Lambda^+}^s(x, b; p, z) = \frac{\bar{k}}{\pi} \int_{\mathbb{R}^2} ds dt \Omega_{\Lambda^+}(x, b; s, t) \exp(-2i\bar{k}(q - t)(p - s)). \quad (22)$$

We use $h_{\Lambda^+}^s(x, b; p, z)$, the symbol in the standard calculus, because it is computationally more efficient. Again, the marching computational algorithm takes the same form as the homogeneous medium construction.

This concludes our approach to modelling the two-way wave propagation problem using a one-way marching algorithm.

DISCUSSION

We have introduced one-way methods into an inherently two-way problem. In a generally inhomogeneous medium, even though the source wave is a one-way wave, once it enters an inhomogeneous environment, internal scattering produces a two-way wavefield. Our model describes these complex behaviours through a pseudodifferential operator, which allows us to employ a one-way marching algorithm to numerically solve for the wavefield.

Throughout the theory examples from various fields of physics have guided us. Wavefield splitting, invariant imbedding, and phase space (Weyl pseudodifferential, Fourier integral operator and path integral) methods were used to reformulate the problem in terms of an operator that includes all scattering physics. There are both physically insightful and mathematically succinct ways to derive governing equations for this scattering operator. We have chosen to present this material as reasonable generalizations of the easily derived results for a homogeneous medium. The results, in effect, provide an extension of Fourier methods to inhomogeneous environments.

The methods and algorithms, especially the uniform asymptotic expansion of the DtN operator, mentioned in this report, are broadly applicable in many different areas of industry as outlined previously.

The developments in this paper provide the framework for seismic depth migration imaging. The uniform asymptotic approximations of the operator symbols extend the well known GPSPI algorithm. Figure 4 illustrates the Marmousi image produced with the leading-order square root function term alone (GPSPI). The inclusion of the uniform terms implied in (14) will result in even greater resolution.

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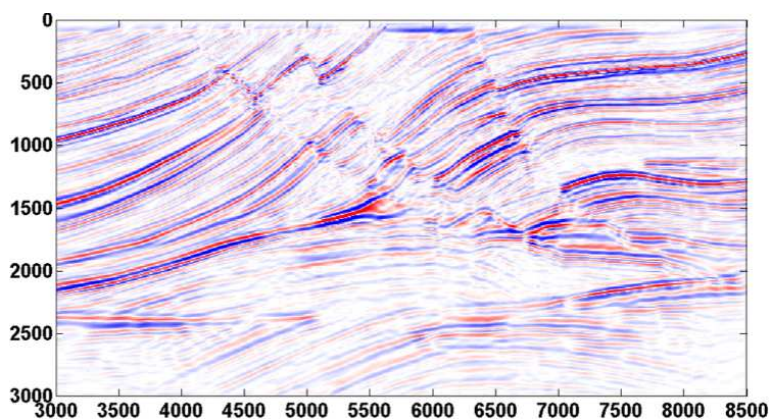


FIG. 4. Image of Marmousi model computed using the leading term of the asymptotic approximation. Hugh Geiger, personal communication.

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