Stability of time variant filters

Michael P. Lamoureux*, Safa Ismail*, and Gary F. Margrave

ABSTRACT

We report on a project to investigate the stability and boundedness properties of interpolated filters that arise in the design of time variant filters. We observe that IIR filters maintain stability under linear interpolation of their PARCOR coefficients, while FIR filters maintain boundedness under linear interpolation of their direct form coefficients. More accurate interpolation between distant filters require higher order interpolation to be effective.

INTRODUCTION

Many important steps in seismic data processing involve applying a filter to data in order to enhance certain signal features, minimize noise, or otherwise alter the signal to a preferred state. Examples include f-k filtering to remove ground roll, spiking deconvolution to resolve reflectivity series, or x-ω wavefield extrapolation for use in migration, among others.

In many cases, a stationary filter is applied to the data in the form of a convolution operator acting on the signal. However, because the signal may not be stationary – for instance, the character of a seismic signal changes as it passes through different geological strata – it can be advantageous to implement a filter that is not stationary. That is, a filter whose characteristics such as frequency and phase response vary over time and/or space. Such a filter is known as a time variant filter, or nonstationary filter.

Our research group in POTSI† has been very active in developing time variant filters for a variety of imaging applications, make use of such techniques as Gabor multipliers, pseudodifferential operators, Fourier integral operators, and adaptive windowing methods to partition a signal into locally stationary parts. A theoretical question that always arises is how does one guarantee that time varying filters designed by these methods have desirable, predictable properties that one is familiar with in stationary filters. That is to say, what can one say about the boundedness, stability, and (local) frequency response of such filters?

These are difficult questions. In this report, we focus on a simpler sub-problem, which is to characterize the stability, boundedness, and response properties of interpolated stationary filters. The motivation is as follows: in a slowly varying filter, one can envision the filter as making a smooth transition from one stationary filter to another. The two stationary filters may have desired, specified characteristics (eg. stability, and a strictly determined frequency response), and the question is whether the filter in-between also has these desired properties. A bad situation would occur if two stable filters suddenly became unstable in the transition from one to the other. The results would be an overall time variant filter that

*Department of Mathematics and Statistics, University of Calgary

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is unstable.

As a simple test of these ideas, we consider two fixed stationary filters H^0 and H^1, and interpolate between them by linearly changing the parameters for one into the parameters of the other. Of course, there are many ways to describe the same filter using different parameter sets (eg: location of roots and zeros, coefficients of the forward and feedback terms, PARCOR coefficients), so we are interested in what happens differently with different choices of parameter sets.

As an illustrative example, we consider interpolating between two Butterworth filters, first using the natural parameters $(\mathbf{b}, \mathbf{a})$ that come up in the direct form of the filter, then using PARCOR coefficients that describe the equivalent lattice filter. As a comparison, we look at what happens with a similar FIR filter. Then we state some theoretical results that suggests higher order interpolations using PARCOR coefficients (the lattice structure form of the filter) is a promising approach to finding intermediate filters with the desirable frequency response.

We are mainly interested in preserving stability and boundedness in the interpolated filter. A good frequency response curve for the interpolated filter is also of interest.

STABILITY VERSUS BOUNDEDNESS

A filter is said to be stable if any bounded data input leads to bounded data output; for stationary filters (eg IIR filters), stability is equivalent to having poles of the filter transfer function located inside the unit circle of the complex plane. Stationary FIR filters are always stable, as they have no poles.

Closely related to stability is the question of boundedness. When a given filter is repeatedly applied to a data sequence, we can ask whether this repetition leads to an output that grows without bound. If it does, then the iterated filter is unstable. Stability of the iterated filter is equivalent to the base filter being stable and bounded by one in amplitude. That is, the gain is never higher than 0dB at any frequency.

In seismic imaging applications, a single filter is often repeated over and over again – wavefield extrapolation is such an instance. It is useful to know when this iterated filter is stable. Therefore, we must ask when the given filter is bounded by one in the amplitude response.

It is possible to confuse stability of the single filter with the stability of the repeated filter. In this paper we will be careful to distinguish between stability and boundedness of the single filter.

THE INTERPOLATED BUTTERWORTH FILTER

The Butterworth filter is a simple IIR filter of interest in seismic imaging as it can be used as an effective lowpass filter used to truncate the evanescent wave energy in the non-propagating region of the f-k plane. Combined with an all-pass filter of the appropriate phase, we obtain a filter suitable for wavefield extrapolation.

A lowpass Butterworth filter is flat in the passband, and in particular it is stable and bounded; the filter response never goes above 0dB. There are explicit design procedures for the Butterworth filter: given an order N and a cutoff frequency f_0, the design procedures produce explicit values for the coefficients $a_0, a_1, \ldots a_N$ and $b_0, b_1, \ldots, b_N$ so that the filter response of the Butterworth filter is given by

$$H(z) = \frac{\sum b_k z^{-k}}{\sum a_k z^{-k}}.$$

Note that usually $a_0 = 1$. These vectors of coefficients $\mathbf{a} = [a_0, a_1, \ldots, a_N]$, $\mathbf{b} = [b_0, b_1, \ldots, b_N]$ are used directly in the transversal (direct form) implementation of an IIR filter, and so they are immediately useful in numerical implementations.

We design a test to observe what happens when interpolating between two selected Butterworth filters. Our specific test is to create two lowpass Butterworth filters H^0 and H^1, with specific cutoff frequencies chosen at 0.3*Nyquis and 0.7*Nyquist. The order is fixed at 10. MATLAB code is used to create corresponding filter coefficient vectors ($\mathbf{a^0}$,$\mathbf{b^0}$) and ($\mathbf{a^1}$,$\mathbf{b^1}$) and interpolate linearly between them. We then examine the properties of the resulting transfer function is for the corresponding interpolated IIR filter.

We plot the results in Figure 1. We observe that the cutoff frequency does move smoothly from 0.3 to 0.7. However, there is a raised lip right at the cutoff, in the interpolated stages. (See the middle three graphs in Figure 1.) This raised lip goes above the 0dB line, indicating that the interpolated filters are not bounded by one. This is not desirable for repeated applications of the filter, which would then become unstable.

The same Butterworth filters may be implemented using a lattice structure. The coefficients describing the lattice filter are called the lattice ($\mathbf{k}$) and ladder ($\mathbf{v}$) coefficients, also known as PARCOR coefficients. The lattice filter has the desirable property that it is stable if and only if these coefficients $(\mathbf{k}, \mathbf{v})$ are bounded by one in absolute value. This has an obvious advantage for interpolation, as stability will be preserved under convex interpolation.

As a second test, we create two lowpass Butterworth filters H^0 and H^1, with specific cutoff frequencies, 0.3*Nyquis and 0.7*Nyquist. The order is fixed at 10. We create corresponding PARCOR coefficients ($\mathbf{k^0}$,$\mathbf{v^0}$) and ($\mathbf{k^1}$,$\mathbf{v^1}$) and interpolate between them, and see what the transfer function is for the corresponding IIR filter.

We plot the results in Figure 2. In this case, there is no lip going above the 0dB line, so these filters are not only stable, but also bounded by one. Thus repeated applications of the filter would remain stable. However, there is a lot of strange behaviour in the filter response at the cutoff region. This is worth further investigation.

FIR EXAMPLE

Since many applications in seismic work with FIR filters, we will do a comparison example to the above two cases, using an FIR lowpass filter. However, we know from theoretical work (later in this paper) that linear interpolations of bounded FIR filters are

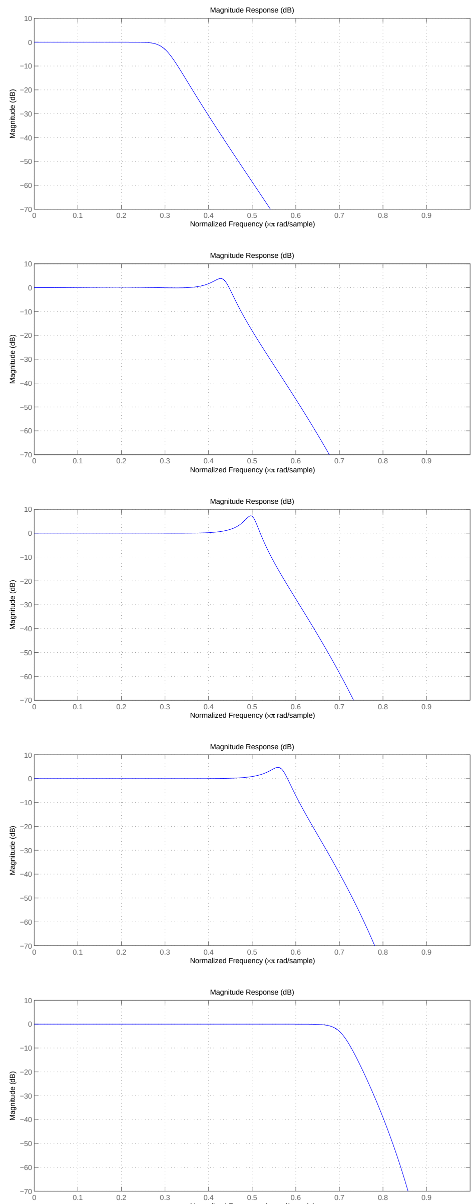

FIG. 1. Interpolation on Butterworth filters with cutoff frequencies 0.3 to 0.7, using direct form parameters b, a. Using convex interpolations with t values of 0, .1, .5, .9, and 1. The raised lip in the three middle figures indicates we exceed the bound of 1 (0dB).

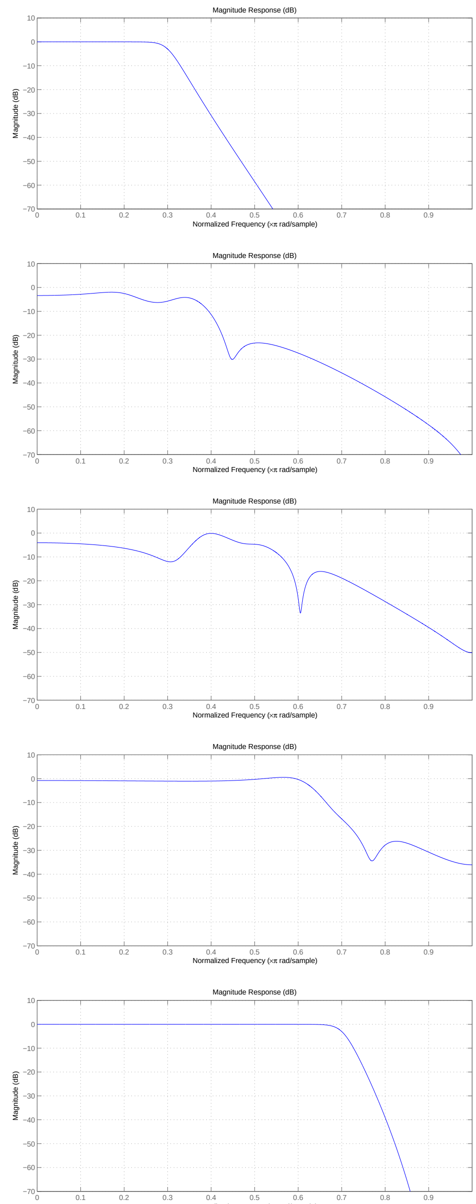

FIG. 2. Interpolation on Butterworth filters with cutoff frequencies 0.3 to 0.7, using lattice form parameters k, v. Using convex interpolations with t values of 0, .1, .5, .9, and 1. The curves all lie underneath the 0dB line, indicating bounded by one. However, the intermediate response curves are not well-behaved.

again bounded. So, all these interpolations will be bounded (by one). And all FIR filters are stable.

Again, we pick cutoff frequencies at 0.3*Nyquist to 0.7*Nyquist, and set the order to 20. (Twice the IIR case, but same number of coefficients.) The results are shown in Figure 3. As expected, the response curve never goes about 0dB, so these interpolated filters are bounded by one.

However, what is new is that we get to see what the filter response of the interpolated filter looks like. As with the lattice case, it does not look too good, as indicated by the middle three plots in Figure 3.

EXAMINING PARAMETER INTERPOLATION

We have been using linear interpolation of coefficients to generate approximations to certain IIR filters, with mixed results. It is worthwhile to examine what the exact parameters should be, and how they actually vary over the range we are trying to interpolate. Perhaps we can do something better than linear approximation.

To this end, we plot exactly the required $(\mathbf{b}, \mathbf{a})$ values for a Butterworth filter, as the cutoff frequency ranges between two frequency values .3 and .7 (of Nyquist). Figure 4 shows plots of this parameter curves. Note they are not linear; however, a quadratic of cubic approximation would do a better job in approximating these exact values. We would expect the resulting interpolated filters to have better cutoff characteristics. We have not yet attempted this approximation.

Figure 5 shows plots of the parameter curves using the lattice filter coefficients $(\mathbf{k}, \mathbf{v})$. Again, these are not linear curves, and we would require even higher order polynomials than cubics to approximate these well. We would expect the resulting interpolated filters to have better cutoff characteristics. We have not yet attempted this approximation.

THEORETICAL RESULTS

We state a few results noting when we can guarantee stability and/or boundedness of a linear interpolation of two filters. These results are reflected in the numerical work above.

Theorem 1 *Suppose H^0 and H^1 are bounded-by-one FIR filters and H^t is the FIR filter obtained by linear interpolation of the direct form coefficients $\mathbf{b}$ of the filters. Then H^t is bounded by one for all $0 \leq t \leq 1$.*

The proof follows from the observation that the filter response of an FIR filter is given by

$$H(z) = \sum b_k z^{-k}$$

and that the filter is bounded by one if any only if $|H(e^{2\pi i f})| \leq 1$ for all points $e^{2\pi i f}$ on the unit circle. The filter response of H^t is just the linear combination of filter response of H^0 and H^1, so

$$H^t(z) = (1-t) * H^0(z) + t * H^1(z)$$

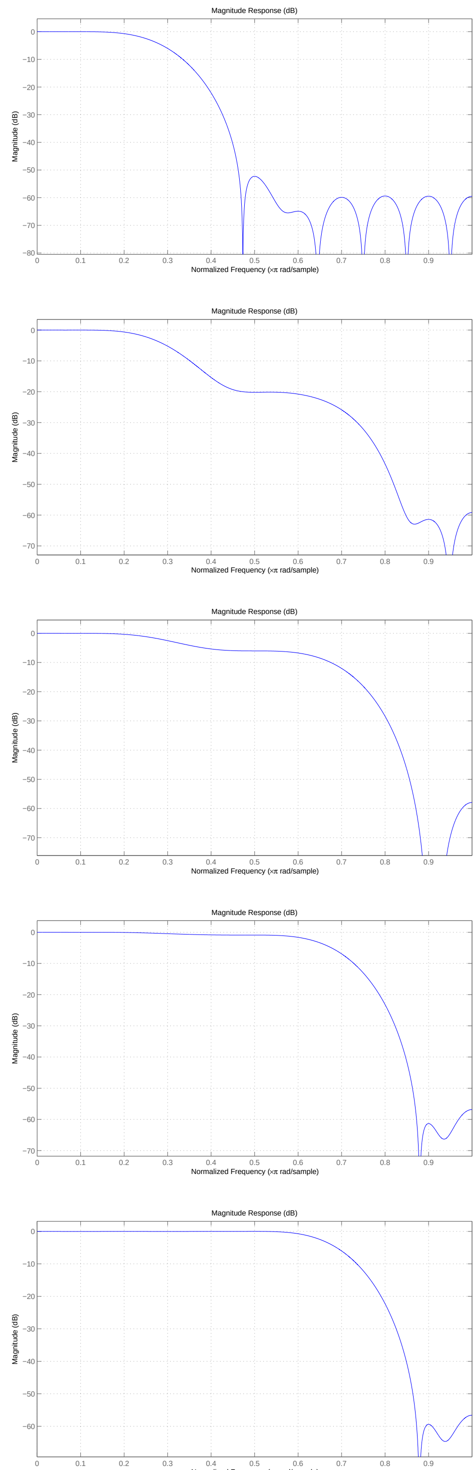

FIG. 3. Interpolation on lowpass FIR filters with cutoff frequencies 0.3 to 0.7, using direct form parameters a. Using convex interpolations with t values of 0, .1, .5, .9, and 1. The curves all lie underneath the 0dB line, indicating boundeness. However, there are other problems.

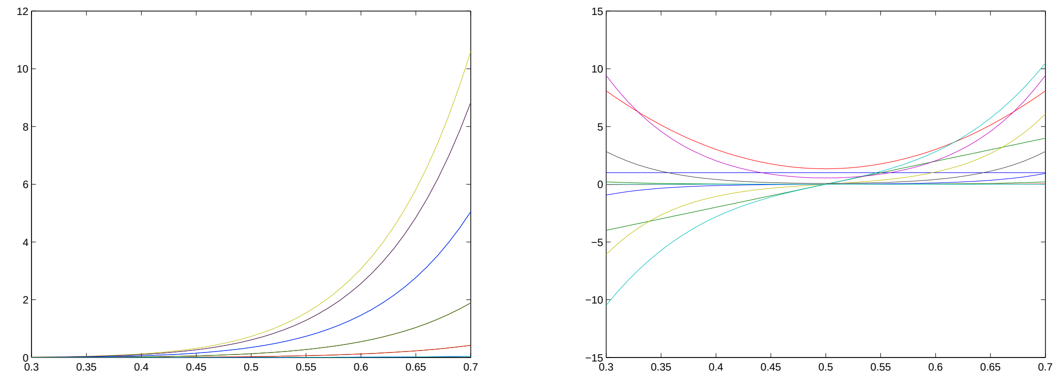

FIG. 4. The (b, a) parameter values for IIR Butterworth filter with cutoff frequency ranging from .3 to .7. Notice these are not linear changes. A cubic interpolation would work much better.

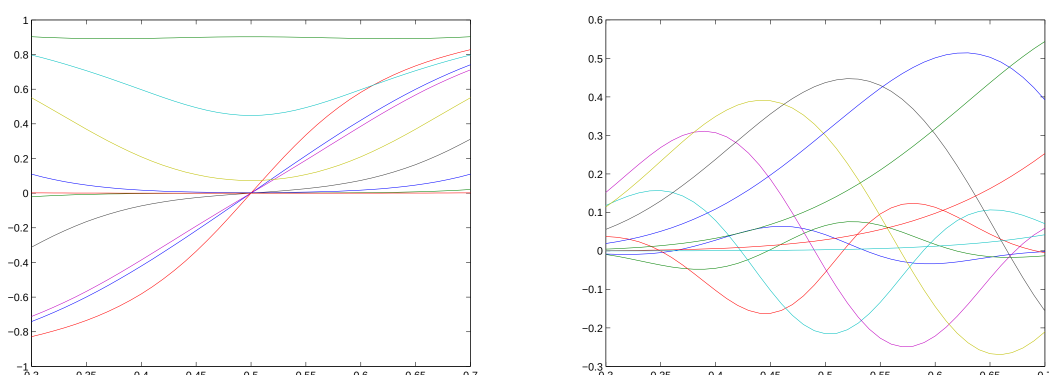

FIG. 5. The (k, v) parameter values for IIR Butterworth filter with cutoff frequency ranging from .3 to .7. Notice these are not linear changes. A higher order approximation is required.

and thus the curve $\{H^t(e^{2\pi i f}) : 0 \leq f \leq 1\}$ is a convex combination of two curves inside the unit disk, and hence is itself inside the unit disk. So $|H^t(e^{2\pi i f})| \leq 1$ for all f, and hence filter H^t is bounded by one.

Note: FIR filters are always stable, as they have no poles. The above result shows repeat application of an interpolated FIR filter remains stable upon repetitions, provided the two initial FIR filters are bounded by one.

Note: IIR filters are not usually bounded-by-one under interpolation because of the coefficients in the denominator of the filter transfer function (which is a rational function, with polynomial factors in both the numerator and denominator). If we held the denominator fixed, then we could interpolate the numerator only, and all would be well. This might be a fruitful approach to the problem.

Another approach is to use IIR filters in the lattice structure form. In this case, the stability (or minimum phase condition) of a filter is reflected in a convex constraint on the lattice form coefficients. Thus linear interpolation will preserve these filter characteristics. However, we don't know yet what happens to boundedness. Although the experimental results above suggest we might have boundedness.

Theorem 2 *Suppose H^0 and H^1 are minimum phase IIR filters and H^t is the IIR filter obtained by interpolating the lattice form coefficients ($\mathbf{k}, \mathbf{v}$) of the filters. Then H^t is minimum phase for all $0 \leq t \leq 1$.*

We don't have a complete proof for this result, but believe it to be true. For an all-pole filter, or an all-zero filter, there is a result in Honig and Messerschmitt (1984) that states the poles (or zeros) of the filter lie inside the unit circle if and only if the lattice coefficients (PARCOR coefficients) satisfy

$$|k_j| \leq 1, \qquad \text{for all indicies } j.$$

This is a convex set, so it is invariant under convex combination. Thus if H^0 and H^1 are minimum phase, then their poles (or zeros) all lie in the unit circle, and thus the k_j are all less than or equal to one in magnitude. The interpolated coefficients will also be less than one in magnitude, and so the interpolated filter H^t also has all its poles (or zeros) inside the unit circle. So the filter H^t is also minimum phase.

This covers the case of the all-pole, and all-zero filters. We require a reference to a result which states for the general IIR filter, implemented as a lattice filter with two sets of coefficients ($\mathbf{k}, \mathbf{v}$) called the lattice and the ladder coefficients, that the $\mathbf{k}$ determine the poles and the $\mathbf{v}$ determine the zeros. The condition for these poles and zeros are, respectively, that $|k_j| \leq 1$ and $|v_j| \leq 1$. These are also convex conditions, so the convex combination of lattice coefficients for minimum phase filters H^0 and H^1 will also be minimum phase.

With this separation of the zeros and poles, we get the following result on stability.

Theorem 3 *Suppose H^0 and H^1 are stable IIR filters and H^t is the IIR filter obtained by interpolating the lattice form coefficients $\mathbf{k}, \mathbf{v}$ of the filters. Then H^t is stable for all $0 \leq t \leq 1$.*

The proof follows by noting that stability for IIR filters means something different: namely, that the poles are all inside the unit circle. As noted above, we believe this is equivalent to the condition that the coefficients k_j are less than or equal to one, in absolute value. There is no constraint on the coefficients v_k. Since this is a convex constraint, the linearly interpolated filter H_t will also as poles inside the unit circle, and hence it too is stable.

We could also compute directly with poles and zeros, and obtain the following theorems. While this is an attractive theoretical results, note that it is in general difficult to numerically determine the poles and zeros of a given filter.

Theorem 4 *Suppose H^0 and H^1 are minimum phase IIR filters and H^t is the IIR filter obtained by linear interpolation of the poles and zeros of the filters. Then H^t is minimum phase for all $0 \leq t \leq 1$.*

Theorem 5 *Suppose H^0 and H^1 are stable IIR filters and H^t is the IIR filter obtained by linear interpolation of the poles and zeros of the filters. Then H^t is stable for all $0 \leq t \leq 1$.*

The proof of the theorems follows by noting that minimum phase (resp. stability) is equivalent to having both poles and zeros (resp. just the poles) inside the unit circle. This is a convex condition, respected by the convex combination of poles and zeros. So filter H^t has the same phase (resp. stability) properties as H^0, H^1.

In these results, there is nothing special about linear interpolation, or two filters H^0, H^1. By the same proof, it is true that any convex combination of FIR direct coefficients, or of IIR PARCOR coefficients, from a family of stable filters, will again be stable.

Unfortunately we have no concrete results concerning the boundedness of the interpolated IIR filter.

GENERAL OBSERVATIONS

FIR filters are always stable, and preserve bounded-by-one under linear interpolation, or more generally, under convex combinations. Thus the interpolated FIR filters will remain stable under repeated application.

IIR filters do not preserve stability, unless we use lattice form IIR filters (interpolation of PARCOR coefficients). The equivalent condition is that the lattice coefficients satisfy $|k_j| \leq 1$. Similar results hold for the minimum phase condition for lattice filters.

We have given a rough proof to show lattice filters preserve stability (or minimum phase) under interpolation; a few details need to be checked. However, other artifacts in the filter response curve are introduced that are not desirable. Interpolation of higher order would give better results. The convex criterion for stability of lattice filters can be incorporated into the interpolation (eg. truncating the $(\mathbf{k}, \mathbf{v})$ parameters to live in the box $|k_j| \leq 1$.)

We don't yet have a result about boundedness for IIR filters in lattice form. We have examples to show IIR filters in direct form can lead to interpolated filters that exceed the bounded-by-one condition (and hence will lead to instability upon repeated application). We have similar examples with lattice filters that seem not to violate the bound-by-one.

More examples would proved useful illustrations of these phenomena. For instance, higher order interpolations of filter coefficients should give better behaviour. We could consider interpolations between filters of different order. Interpolations on all-pass filter used in seismic wavefield extrapolations would be interesting and useful. We are working on these examples.

Note that in seismic application, minimum phase filters are often designed using the Levinson-Durbin algorithm. This algorithm also computes the PARCOR coefficients as a matter of course, so we can use existing algorithms to find the lattice filter parameters. We have not yet done this calculation on a real seismic filter.

COMPUTATIONAL ISSUES

We use the Signal Processing toolkit in MATLAB to do all the filter computations. The 'butter' command creates the coefficients $(\mathbf{b}, \mathbf{a})$ of a Butterworth filter. Interpolating on the vectors $(\mathbf{b}, \mathbf{a})$ gives the interpolated filters. The 'dfilt.df2' command creates a direct form Type II filter from these parameters, and the 'convert' creates the equivalent lattice filter. We use the latticearma conversion to create an autoregressive moving average lattice filter, which can simulate the IIR Butterworth filter. Lattice parameters $(\mathbf{k}, \mathbf{v})$ specify the lattice filter, and we interpolate on the vectors $(\mathbf{k}, \mathbf{v})$ to get the interpolated lattice filters. The command 'fvtool' is used to view the frequency response of each filter.

SUMMARY

We have given several examples to illustrate how a time variant filter might behave, by modeling such a filter with interpolations between stationary (time invariant) filters. We demonstrate that care must be taken to ensure stability, boundedness, and acceptable filter response in these interpolated filters.

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