

Sensitivity measurements for locating microseismic events

John C. Bancroft, Joe Wong, and Lejia Han

ABSTRACT

The first arrival clock-times from a number of receivers are used to estimate the clock-time and location of a microseismic event. The 3D solution is based on a 2D source location scheme based on the Apollonius method. This 3D solution requires four receivers that are non-coplanar or non-collinear. These restrictions are typically violated when receivers are placed in a large grid on the surface or in a linear array in a well. Non Apollonius solutions are presented for these restricted cases along with an analysis that relates the accuracy of the estimated source location to clock-times at the receivers.

It is anticipated that these methods will be part of a larger system where these analytic solutions can be used with data extracted from the large arrays.

INTRODUCTION

The problem of identifying the location of a microseismic event has application to well fracing material or CO₂ sequestration, or the location of impending geological hazards such as landslides or major earthquakes. Other areas for applications of these techniques are converting raypath traveltimes to gridded traveltimes, sniper locating, or global positioning.

Microseismic events may be located by a number of techniques that include a three component receiver that uses the three components to estimate the arrival direction of the wavefield, then use the difference in P and S-wave arrival times to estimate a distance. Other techniques use first arrival clock-times, then “search over a grid of hypothesized source locations”, (Daku et al. 2004) or the wavefield propagation (seismic migration) of many surface receivers (Chambers et al. 2008).

The approach in this paper uses the first-arrival clock-times from a number of receivers to estimate the clock-time of the event and to estimate its location. Earlier papers have shown how the 2D problem of locating the source can be solved by constructing a circle tangent to three other circles. These circles were centered at the receiver location and had a radius proportional to the clock-times of three non-collinear receivers. The construction of a tangent circle to three other circles was solved by Apollonius about 200 years B.C., and many algebraic solutions based on the geometrical solution have since been derived.

The 3D solution to the problem that is based on the 2D solution that allowed the estimation of the clock-time and location of the source (Bancroft and Du 2006 and 2007). The 3D solution required the construction of a sphere that was tangent to four other spheres, with restrictions that the receivers locations be non-coplanar or non-collinear. We present solutions for these cases where four receivers are co-planar on a square grid on the surface, or three equally spaced collinear receivers in a vertical well. The solution for three collinear receivers becomes a 2D problem and is only able to solve the depth

and radial distance of the source from the well. The three collinear receivers cannot estimate the azimuth.

The sensitivity of Apollonius, square grid, and three vertical receiver methods will be discussed relative to the accuracy of the receiver clock-times. We assume a constant or RMS type velocity.

The first arrival clock-times are only relative to the source clock-time and would be very large values. Any arbitrary time could be added or subtracted to these times and will be independent of the actual source location. In practice, I subtract the minimum receiver clock-time from all the receiver clock-times. The source clock-time will then be negative, an aid in simplifying the choice of the correct Apollonius solution.

THEORY

The Apollonius solutions are based on geometric constructions and have relatively simple algebraic solutions (+, -, * and /) and require only one square-root. However, part of the computation requires a division that can go to zero if the receivers are co-planar or co-linear. We are able to overcome the Apollonius restriction for two specific geometries; one for a planar surface with four receivers on a square grid, and the other for three equally spaced co-linear receivers. The four receivers on a square grid may be applicable to a large array of receivers on the surface, while the three collinear receivers are applicable to receivers in a well.

The traveltimes equations for raypaths between a source at (x_0, y_0, z_0) and four arbitrarily located receivers at (x_1, y_1, z_1) , (x_2, y_2, z_2) , (x_3, y_3, z_3) , and (x_4, y_4, z_4) are:

$$\begin{aligned} (x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2 &= v^2 (t_1 - t_0)^2 \\ (x_2 - x_0)^2 + (y_2 - y_0)^2 + (z_2 - z_0)^2 &= v^2 (t_2 - t_0)^2 \\ (x_3 - x_0)^2 + (y_3 - y_0)^2 + (z_3 - z_0)^2 &= v^2 (t_3 - t_0)^2 \\ (x_4 - x_0)^2 + (y_4 - y_0)^2 + (z_4 - z_0)^2 &= v^2 (t_4 - t_0)^2, \end{aligned} \tag{1}$$

where v is the (constant) velocity, t_0 is the clock-time of the source event and t_1, t_2, t_3 , and t_4 , are the clock-times of the event at the corresponding receivers. These equations are the starting point of the Apollonius solution (Bancroft and Du 2006 and 2007).

Four coplanar receivers on a square grid

We now restrict the four receivers to be located on a square grid with one point at the origin $(0, 0, 0)$ and the other three points separated by the distance h , i.e. $(h, 0, 0)$, $(0, h, 0)$ and $(h, h, 0)$ all located on the surface with $z_i=1$ to 4. This simplification allows us to define the four equations as:

$$\begin{aligned} x_0^2 + y_0^2 + z_0^2 &= v^2 (t_1 - t_0)^2 \\ (x_0 - h)^2 + y_0^2 + z_0^2 &= v^2 (t_2 - t_0)^2 \\ x_0^2 + (y_0 - h)^2 + z_0^2 &= v^2 (t_3 - t_0)^2 \\ (x_0 - h)^2 + (y_0 - h)^2 + z_0^2 &= v^2 (t_4 - t_0)^2, \end{aligned} \tag{2}$$

This simplification to the raypath equations bypasses the divide-by-zero restriction of the more general solution for planar receivers, and algebraic manipulation allows the definition of the source clock-time t_0 to be:

$$t_0 = \frac{t_1^2 - t_2^2 - t_3^2 + t_4^2}{2(t_1 - t_2 - t_3 + t_4)} \quad (3)$$

This solution for t_0 is independent of the size of the square at h and the velocity v . The source coordinates x_0 , y_0 , and z_0 are given by:

$$\begin{aligned} x_0 &= \frac{v^2 \left[2t_0(t_2 - t_1) - (t_2^2 - t_1^2) \right] + h^2}{2h} \\ y_0 &= \frac{v^2 \left[2t_0(t_3 - t_1) - (t_3^2 - t_1^2) \right] + h^2}{2h} \\ z_0 &= \pm \text{sqrt} \left[v^2 (t_1 - t_0)^2 - (x^2 + y^2) \right] \end{aligned} \quad (4)$$

The depth or z_0 location is computed using a square-root providing two possible locations that are either above or below the surface, simplifying the solution. The derivations of equations (3) and (4) are in Appendix 1

Three collinear receivers

The symmetry of first arrival clock-times of a vertical array of receivers will not allow an estimation of the source azimuth. Consequently the problem is reduced to a 2D problem with three vertical receivers in an (x, z) plane. When the three receivers are separated by distance h , the raypath equations become:

$$\begin{aligned} (x_1 - x_0)^2 + (z_1 - z_0)^2 &= v^2 (t_1 - t_0)^2 \\ (x_1 - x_0)^2 + [(z_1 + h) - z_0]^2 &= v^2 (t_2 - t_0)^2 \\ (x_2 - x_0)^2 + [(z_1 + 2h) - z_0]^2 &= v^2 (t_3 - t_0)^2 \end{aligned} \quad (5)$$

If we take the traveltimes between two adjacent receivers to be $t_h = h/v$, then the micro-source clock-time is given by:

$$t_0 = \frac{t_1^2 - 2t_2^2 + t_3^2 - 2t_h^2}{2(t_1 - 2t_2 + t_3)} \quad (6)$$

The solution is in cylindrical coordinates where the radial distance r_0 replaces x_0 , and the depth of the micro-source, z_0 , are given by:

$$\begin{aligned} r_0 &= \text{sqrt} \left[v^2 (t_1 - t_0)^2 - (z_1 - z_0)^2 \right] \\ z_0 &= \pm \text{sqrt} \left[v^2 (t_1 - t_0)^2 - (x^2 + y^2) \right] \end{aligned} \quad (7)$$

The three collinear receivers in a vertical well cannot establish an azimuth for the source location. However, receivers in a deviated well may not be collinear or coplanar and allow the use of the conventional Apollonius solution to identify all component of the source (x_0, y_0, z_0). The derivation of equations (6) and (7) are in Appendix 2.

EXAMPLES

Apollonius solution

The first example shows the Apollonius solution for four receivers arbitrarily located near the surface, and the source located at a depth that matches the spread of the receivers. When there are no errors, the source is located within the accuracy of the computer and any error is imperceptible. A source clock-time was chosen, and the clock-times at the receivers calculated. Gaussian random noise was added to the clock-time of the receivers, then, using only the receiver locations and their clock-times, a source location and its clock-time were estimated. This procedure was repeated one-hundred times and the mean and standard variation of the source location estimated. Figure 1 shows four views, (side from y, side from x, plan, and perspective), with the receivers in green “x” symbols, a the source location by a blue “+” symbol.

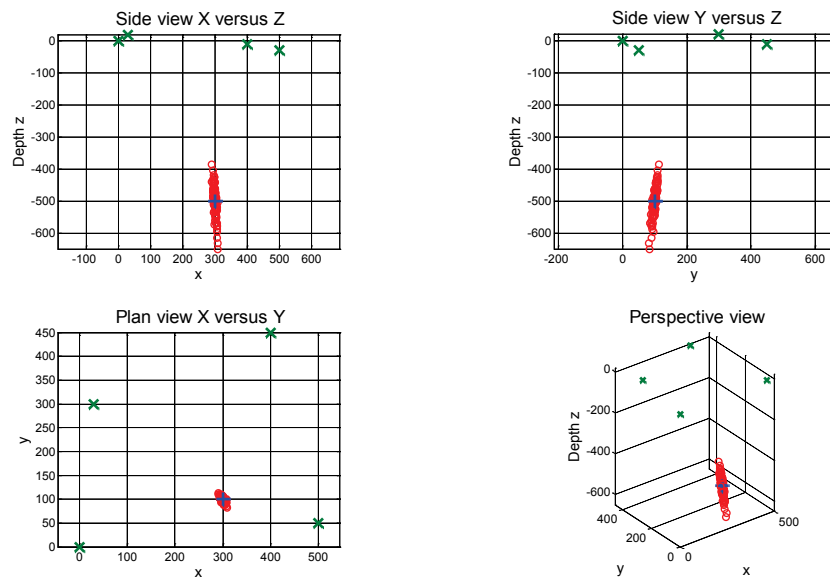


FIG. 1 Four views of an Apollonius solution with four receivers “x” near the surface, the known source “+” and the estimated source locations with a red “o”. The standard deviation of the noise was 1 ms.

The results for the above figure are:

```
x0 = 300.00  y0 = 100.000000.2  z0 = -500.00
xMn = 299.89  yMn = 99.856575.2  zMn = -500.76
xStd = 4.02  yStd = 6.195285.2  zStd = 50.07
(TestensitivityFourArbLocReceiversMicroSeis.m)
```

where x_0, y_0 and z_0 are the defined locations, $xMn, yMn,$ and zMn are the mean locations of the 100 trials, and $xStd, yStd,$ and $zStd$ are the standard deviations of the estimated source locations.

Nest the source location is moved a significant distance to $x_0 = 2000$ m yielding the results displayed in Figure 2. The spread of the estimated source locations has increased.

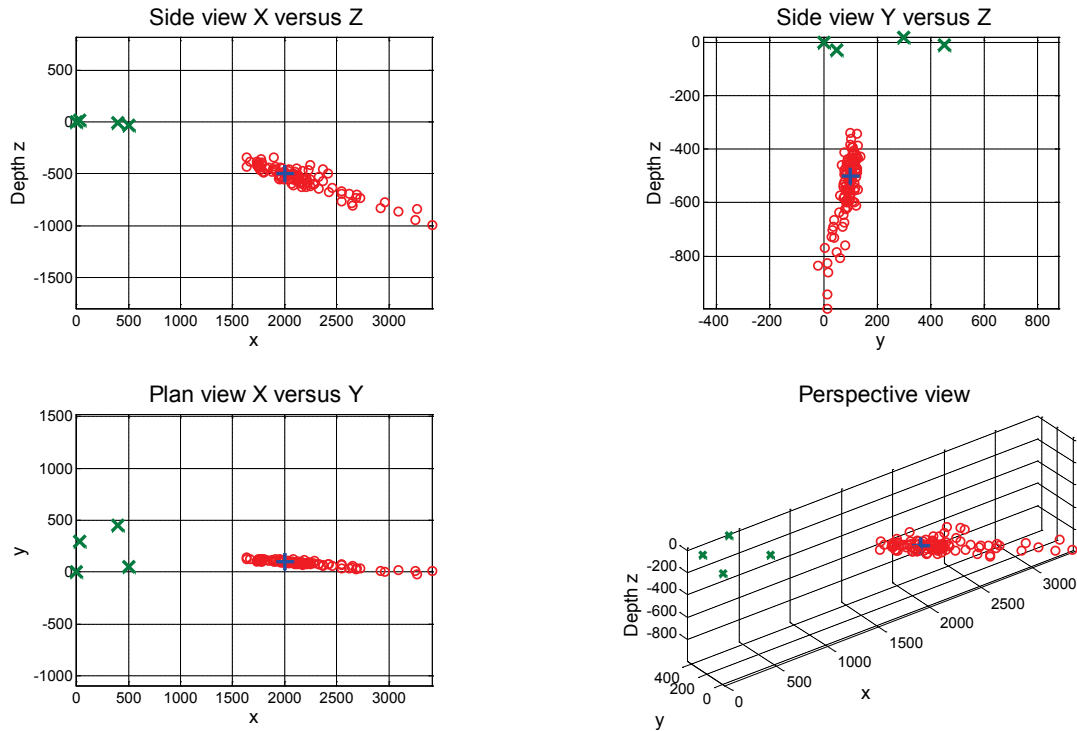


FIG. 2 Similar to Figure 1, however, the source has been moved to $x = 2000$ m.

The parameters for the trials in Figure 2 follow:

```

Model velocity = 2000  t0 = 0
Defined source values t0 = 0
x0 = 2000  y0 = 100  z0 = -500
x1 = 0  y1 = 0  z1 = 0
x2 = 500.05  y2 = 50  z2 = -30
x3 = 30  y3 = 300  z3 = 20
x4 = 400  y4 = 450  z4 = -10
Delta times  t1 = 1.032  t2 = 0.78633  t3 = 1.0236  t4 = 0.85478  Mean = 0.92418
Clock times  t1 = 1.032  t2 = 0.78633  t3 = 1.0236  t4 = 0.85478  Mean = 0.92418
Minimum clock time tmin = 0.78633
Adjusted times t1 = 0.24566  t2 = 0  t3 = 0.2373  t4 = 0.068452  Mean = 0.13785

x0 = 2000.00  y0 = 100.000000.2  z0 = -500.00
xMn = 2147.06  yMn = 89.243578.2  zMn = -545.99
xStd = 363.21  yStd = 30.858586.2  zStd = 128.15
    
```

The above results are misleading as the actual distribution of the sources is much narrower than the standard deviations would indicate. A more accurate description of the distributed locations would be along the estimated sources.

The accuracy of the estimated source location is dependent on its location relative to the receivers. I computed the error of the estimated source depth for many x and y locations on a grid. At each grid point I computed 100 trials, each with different noise. The standard deviation of the noise was 1 ms. The receiver locations were the same as those in the previous example, and the source depth was maintained at 500 m. An alpha-

trim mean was included to remove extreme errors in the estimates of the 100 trials. The trim was 10% off the top and bottom of the sorted estimates. The result of the errors in the vertical components are displayed in Figure 3, which also includes four vertical black lines that represent the location of the four receivers. Some errors in this figure are much larger and are clipped at the maximum values.

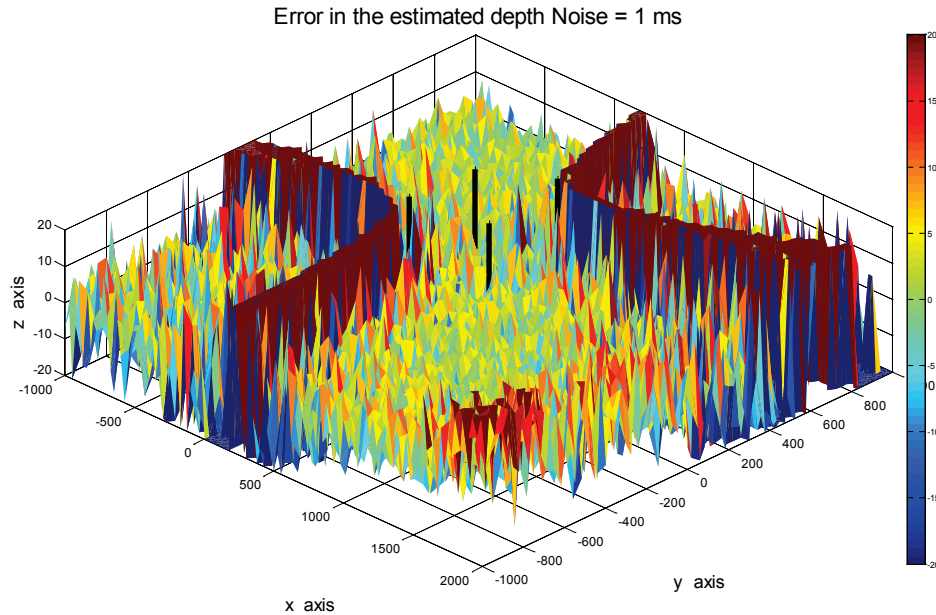


FIG. 3 Plot of the vertical error in meters for the same receiver locations above, with the source located at the same depth, but on a grid in x and y. The depth of the source is 500 m.

The receiver locations for the data in Figure 3 are:

x1 =	0	y1 =	0	z1 =	0
x2 =	500	y2 =	50	z2 =	-30
x3 =	30	y3 =	300	z3 =	20
x4 =	400	y4 =	450	z4 =	-10

The y location of the third receiver was moved from $y_3 = 300$ m to $y_3 = 600$ m :

x1 =	0	y1 =	0	z1 =	0
x2 =	500	y2 =	50	z2 =	-30
x3 =	30	y3 =	600	z3 =	20
x4 =	400	y4 =	450	z4 =	-10

producing the significant change in the error pattern of Figure 4.

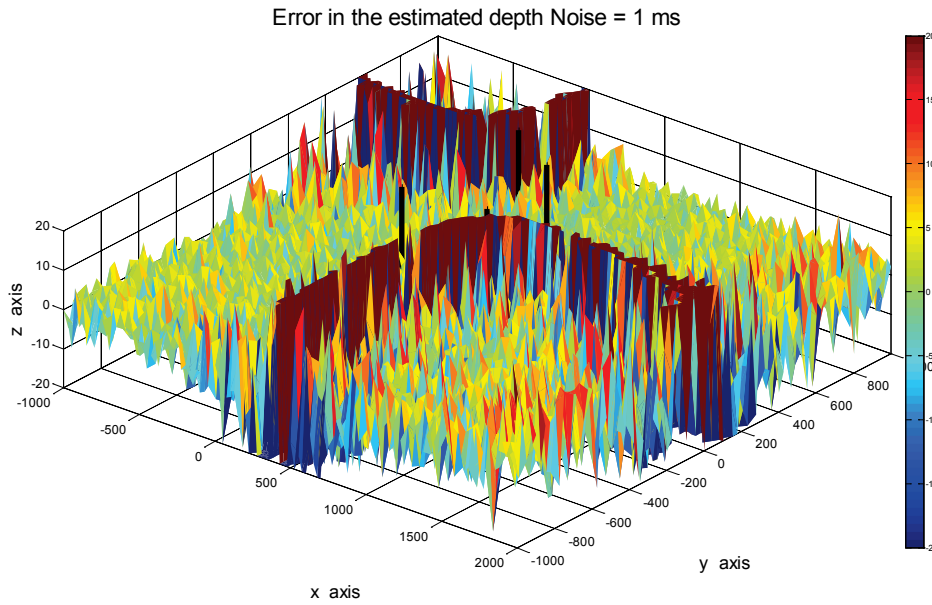


FIG. 4 Plot of the vertical error when the x component of the third receiver was moved from 300 m to 600 m.

Other tests indicate that the areas with poor accuracy appear to be more dependent on the x , and y locations of the receivers, and less dependent on the depth z , of the receivers.

Four coplanar receivers on a square grid

Figure 5 shows the distribution of estimated source locations using four coplanar receivers on a square grid. Noise with a SD of 1.0 ms was added to the clock-times.

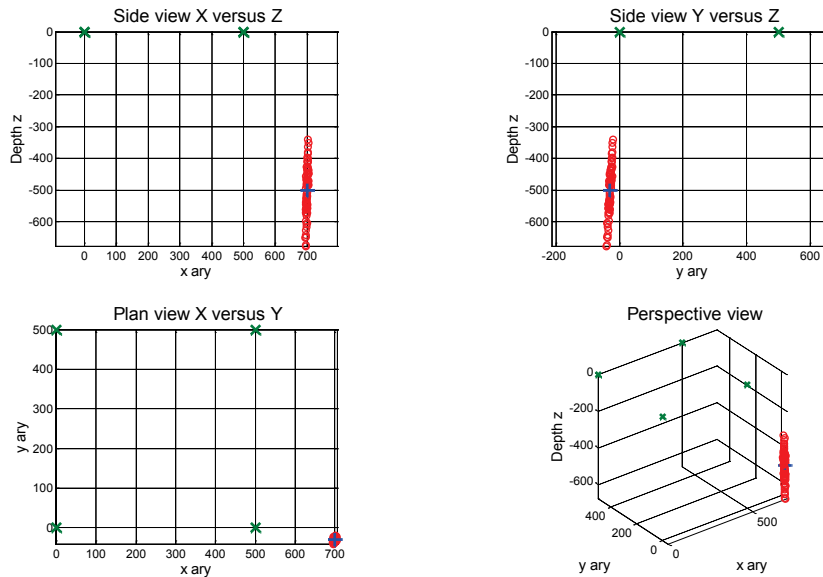


FIG. 5: Four views of estimated source solutions “o” when using a coplanar square array of four receivers “x” on the surface with a defined source “+”.

Data were created similar to the previous example but used a square grid with a distance of 500 m between the receivers. One hundred trials were created and Gaussian noise, with a standard deviation of 1 ms, was added to the receiver times. The results are shown in Figure 5.

The geometry and results for the data in Figure 5 are:

```

x1 = 0   y1 = 0   z1 = 0
x2 = 500 y2 = 0   z2 = 0
x3 = 0   y3 = 500 z3 = 0
x4 = 500 y4 = 500 z4 = 0

x0 = 700.00  y0 = -30.00  z0 = -500.00
xMn = 699.55  yMn = -29.76  zMn = -501.47
xStd = 2.52   yStd = 4.55   zStd = 76.24

```

The source was then spread over a large grid as in the examples of Figures 3 and 4, and the vertical component of the source location displayed.

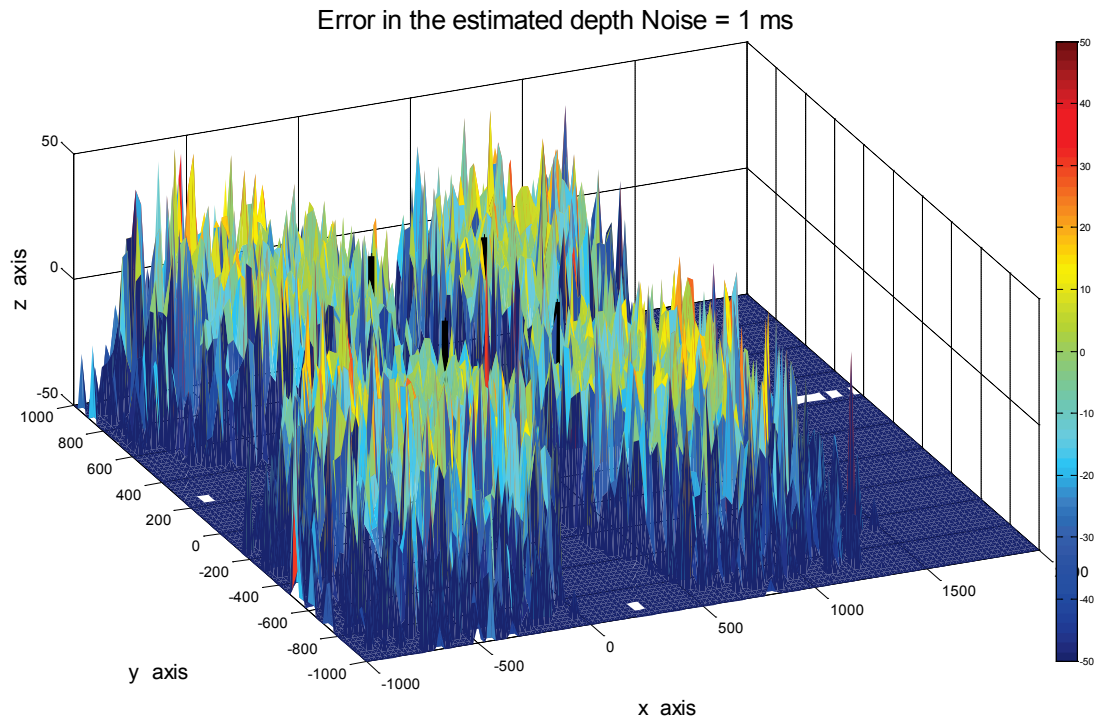


FIG. 6 Error in the estimation of source location using a square grid. The standard deviation of the noise is 1 ms.

The error in the estimation of the source varies considerably, however, there are areas where the noise is reasonably behaved, but there are also areas where the noise tends to extreme values. The following figure reduced the noise to a standard deviation to 0.1 ms, i.e. reducing the noise by a factor of 10. The error in the source estimate is considerably reduced and valid results cover a much larger area.

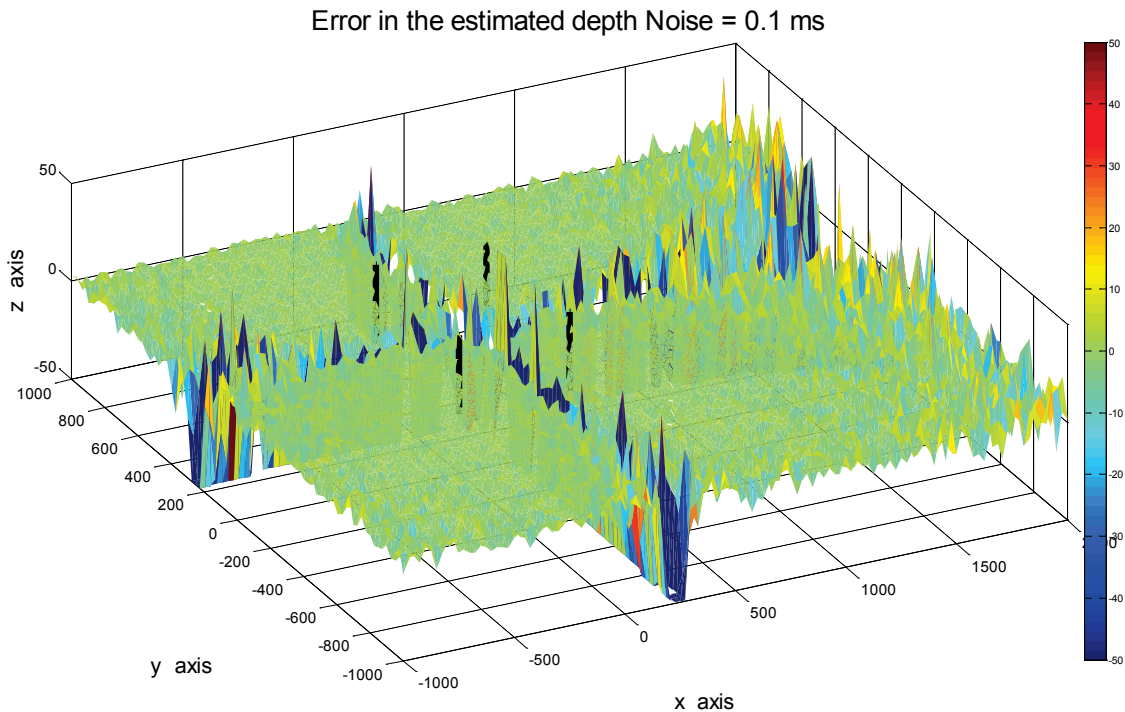


FIG. 7 Error in the estimation of source location using a square grid. The standard deviation of the noise in 0.1 ms.

Three receivers in a vertical array

A simulation of three equally spaced and collinear receivers was conducted. The results are contained in a 2D plane, and if the receivers are in a vertical well, only the depth and radial distance from the well can be estimated. A source location was defined and the traveltimes to the three receivers estimated. Using only the receiver geometry and the three receiver clock-times, the source location was estimated. One hundred trials were conducted with noise added to the receiver clock-times. The results of one example are shown in Figure 8.

Some of the trials failed to produce a location, and the spread was large for a noise of 1 ms. An alpha-trim mean (10%) was once again applied. The alpha-trim mean identified 6 failed computations in 100 tests. The trim size on the remaining samples was 9, and after the 100 estimated depth values we sorted, samples from 10 to 85 were used. The results of the data without and then with the alpha-trim mean are displayed in Figure 8. The six failed computations are also not included in the display without the Alpha-trim mean. The Black “+” is the estimated location of the source for the corresponding test.

The noise in Figure 8 (1.0 ms) may be relatively large so the following figure is included to show the results of relatively small noise on 0.1 ms added to the receiver clock-times. The distribution of the estimated locations is much smaller in Figure 9, and there were no failed estimates.

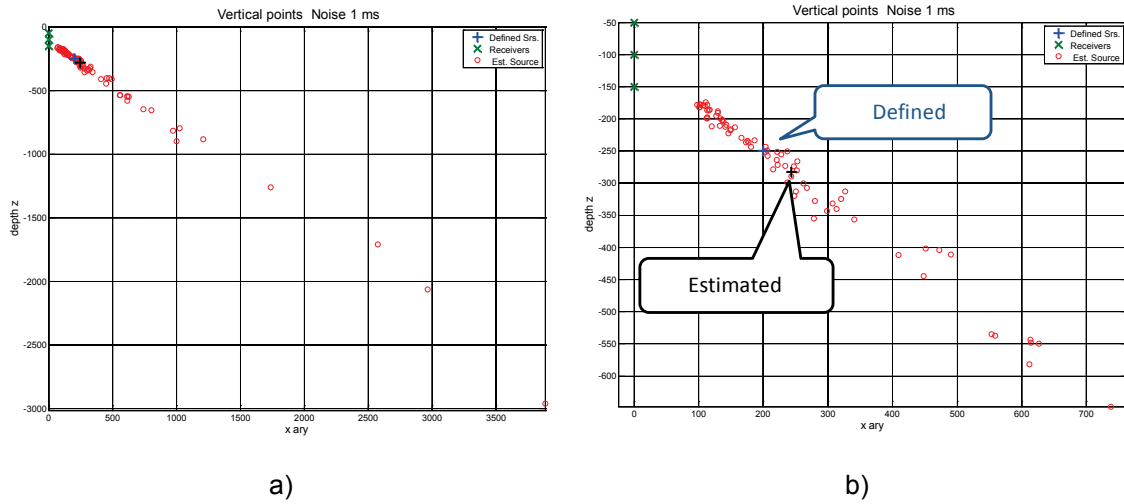


FIG. 8 Three vertical receivers “x” in a simulated well with a defined source “+” and many estimated source locations “o”, a) all usable data and b) with an alpha-trim mean applied. The noise on the receiver clock-times had a standard deviation of 1.0 ms.

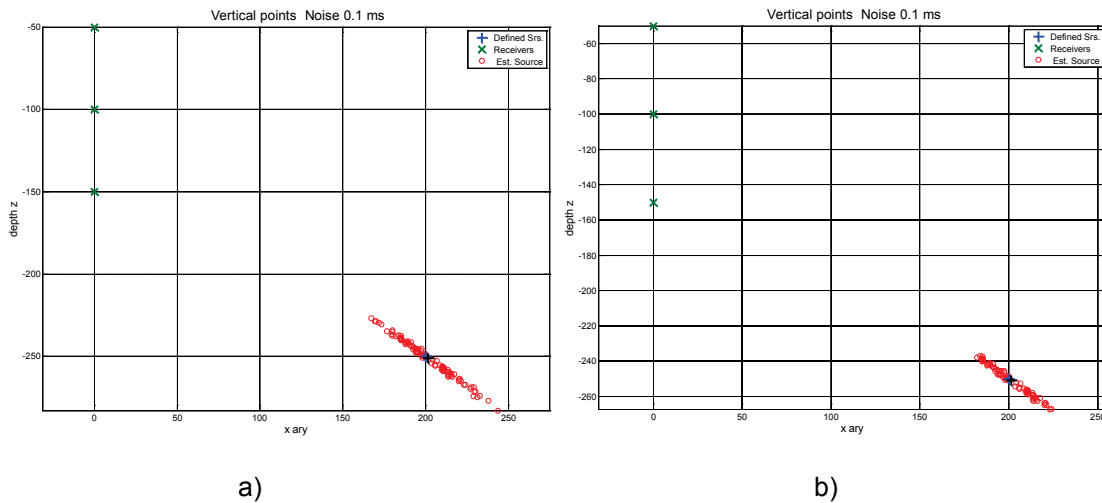


FIG. 9 Three vertical receivers “x” in a simulated well with a defined source “+” and many estimated source locations “o”, a) all usable data and b) with an alpha-trim mean applied. The noise on the receiver clock-times had a standard deviation of 0.1 ms.

The examples in Figures 8 and 9 use three equally spaced collinear receivers. Many combinations of three equally spaced receivers can be extracted from a much larger array of vertical receivers. Each combination can yield an estimated source location. The mean of all these estimated sources will produce an estimated source location for the entire array. A companion paper will illustrate this process.

COMMENTS AND CONCLUSIONS

Three methods for computing the location of microseismic events were presented. Each method only uses the location and first arrival clock-times of the receivers. The first method was based on the Apollonius solution and used four receivers in an arbitrary configuration. This method would fail if the receivers were collinear or coplanar, so two additional methods were presented; four coplanar receivers on a square grid, and three collinear, equally spaced, receivers. The medium was assumed to have a constant velocity, or a geometry where RMS type velocities could be used. Each method had an analytic solution that returned the exact solution in the absence of noise.

Typical microseismic project use large arrays of receivers. Statistical methods are typically used to solve for an estimated source location. It is assumed that these methods would be used as part of a much bigger process where selected receivers are used to provide initial source locations. A typical application is included in a companion paper in which many combinations of three equally spaced receivers are extracted from a vertical array of receivers. The source location is estimated from the mean of the analytic solutions.

Noise was added to the receiver clock-times as part of a sensitivity analysis to evaluate the accuracies that can be expected from a geophysical configuration. Distributions of the estimated locations were displayed.

The accuracy of estimated locations varied with the location of the source relative to the receivers and to the distance from the source to the receivers. The accuracy also varied with the amount of noise on the receiver clock-times.

SOFTWARE

The software in this report was developed with MATLAB.

```
\20090Matlab\TraveltimeMicroseismic\  
  TestSensitivityFourArbLocReceiversMicroSeis.m  
  TestSensitivityFourArbLocReceiversMicroSeisGrdSrs.m  
  TestSensitivityFourArbLocReceiversMicroSeisOneSrs.m  
  TestSensitivityFourReceiversOnSquare.m  
  TestSensitivityFourReceiversOnSquareGrdSRS.m  
  TestSensitivityThreeInLineReceiversBasic.m
```

ACKNOWLEDGEMENTS

We wish to thank NSERC and the sponsors of the CREWES consortium for supporting this research.

CREWES: The Consortium for Research in Elastic Wave Exploration Seismology

REFERENCES

- Bancroft, J.C., 2007, Visualization of spherical tangency solutions for locating a source point from the clock time at four receiver locations, CREWES Research Report Vol. 19, Ch. 44.
- Bancroft, J.C. and Du, X., 2007, Traveltime computations for locating the source of micro seismic events and for forming gridded traveltime maps, 69th EAGE Conference, London, England
- Bancroft, J.C. and Du, X., 2007b, Traveltime computations for locating the source of micro seismic events and for forming gridded traveltime maps, CSPG CSEG Convention, Calgary, Alberta, Canada.
- Bancroft, John C., 2006, Locating microseismic events and traveltime mapping using locally spherical wavefronts, CREWES Research Report
- John C. Bancroft, and Xiang Du, 2006, The computations of traveltimes when assuming locally circular or spherical wavefronts, SEG International Convention, New Orleans

APPENDIX

Appendix 1 Four coplanar receiver on a square grid

Assume four receivers are on the surface, i.e. and $z_1 = z_2 = z_3 = z_4 = 0$, and the four corner points are $(0,0,0)$, $(h,0,0)$, $(0,h,0)$ and $(h,h,0)$. The traveltime equations are:

$$\begin{aligned}
 x_0^2 + y_0^2 + z_0^2 &= v^2 (t_1 - t_0)^2 \\
 (x_0 - h)^2 + y_0^2 + z_0^2 &= v^2 (t_2 - t_0)^2 \\
 x_0^2 + (y_0 - h)^2 + z_0^2 &= v^2 (t_3 - t_0)^2 \\
 (x_0 - h)^2 + (y_0 - h)^2 + z_0^2 &= v^2 (t_4 - t_0)^2
 \end{aligned} \tag{8}$$

Subtract the first equation from the remaining three to get:

$$\begin{aligned}
 -2h_0x_0 + h^2 &= v^2 (t_2 - t_0)^2 - v^2 (t_1 - t_0)^2 \\
 -2h_0y_0 + h^2 &= v^2 (t_3 - t_0)^2 - v^2 (t_1 - t_0)^2 \\
 -2hx_0 - 2hy_0 + 2h^2 &= v^2 (t_4 - t_0)^2 - v^2 (t_1 - t_0)^2
 \end{aligned} \tag{9}$$

Subtracting the first and second line from the third will zero the LHS, and the velocities will divide out, giving only equations in time:

$$0 = (t_4 - t_0)^2 - (t_1 - t_0)^2 - (t_3 - t_0)^2 + (t_1 - t_0)^2 - (t_2 - t_0)^2 + (t_1 - t_0)^2 \tag{10}$$

Squaring and cancelling terms we get:

$$0 = t_1^2 - t_2^2 - t_3^2 + t_4^2 - 2t_0t_1 + 2t_0t_2 + 2t_0t_3 - 2t_0t_4 \tag{11}$$

Solving for t_0 ,

$$2t_0(t_1 - t_2 - t_3 + t_4) = t_1^2 - t_2^2 - t_3^2 + t_4^2 \tag{12}$$

or

$$t_0 = \frac{t_1^2 - t_2^2 - t_3^2 + t_4^2}{2(t_1 - t_2 - t_3 + t_4)} \quad (13)$$

From the first line in equation (9)

$$-2h_0 x_0 + h^2 = v^2 (t_2 - t_0)^2 - v^2 (t_1 - t_0)^2 \quad (14)$$

we solve for x_0 ,

$$x_0 = \frac{v^2 [(t_1 - t_0)^2 - (t_2 - t_0)^2] + h^2}{2h} \quad (15)$$

and from the third line we get

$$y_0 = \frac{v^2 [(t_1 - t_0)^2 - (t_3 - t_0)^2] + h^2}{2h} \quad (16)$$

The depth z_0 is obtained from the first line in equation (8)

$$x_0^2 + y_0^2 + z_0^2 = v^2 (t_1 - t_0)^2 \quad (17)$$

or

$$z_0 = \text{sqrt} \left(v^2 (t_1 - t_0)^2 - x_0^2 - y_0^2 \right) \quad (18)$$

Appendix 2 Three vertical receivers

Start with three receivers in a 2D plane:

$$\begin{aligned}(x_1 - x_0)^2 + (z_1 - z_0)^2 &= v^2(t_1 - t_0)^2 \\(x_2 - x_0)^2 + (z_2 - z_0)^2 &= v^2(t_2 - t_0)^2 \\(x_3 - x_0)^2 + (z_3 - z_0)^2 &= v^2(t_3 - t_0)^2\end{aligned}\tag{19}$$

We located the vertical array at $x = 0$, making $x_1 = x_2 = x_3 = 0$. The equal spacing between receivers is h , making, $z_2 = z_1 + h$, and $z_3 = z_1 + 2h$, giving:

$$\begin{aligned}x_0^2 + (z_1 - z_0)^2 &= v^2(t_1 - t_0)^2 \\x_0^2 + (z_1 + h - z_0)^2 &= v^2(t_2 - t_0)^2 \\x_0^2 + (z_1 + 2h - z_0)^2 &= v^2(t_3 - t_0)^2\end{aligned}\tag{20}$$

Subtract the first equation from the next two, to get:

$$\begin{aligned}(z_1 + h - z_0)^2 - (z_1 - z_0)^2 &= v^2(t_2 - t_0)^2 - v^2(t_1 - t_0)^2 \\(z_1 + 2h - z_0)^2 - (z_1 - z_0)^2 &= v^2(t_3 - t_0)^2 - v^2(t_1 - t_0)^2\end{aligned}\tag{21}$$

Expanding and cancelling produces:

$$\begin{aligned}-2hz_0 + h^2 + 2z_1h &= v^2(t_2^2 - 2t_2t_0 - t_1^2 + 2t_1t_0) \\-4hz_0 + 4h^2 + 4z_1h &= v^2(t_3^2 - 2t_3t_0 - t_1^2 + 2t_1t_0)\end{aligned}\tag{22}$$

The first equation is multiplied by two:

$$\begin{aligned}-4hz_0 + 2h^2 + 4z_1h &= 2v^2(t_2^2 - 2t_2t_0 - t_1^2 + 2t_1t_0) \\-4hz_0 + 4h^2 + 4z_1h &= v^2(t_3^2 - 2t_3t_0 - t_1^2 + 2t_1t_0)\end{aligned}\tag{23}$$

and then subtract the first from the second to get:

$$\frac{2h^2}{v^2} = t_3^2 - 2t_3t_0 - t_1^2 + 2t_1t_0 - 2t_2^2 + 4t_2t_0 + 2t_1^2 - 4t_1t_0,\tag{24}$$

which can be simplified and t_0 isolated as:

$$\frac{2h^2}{v^2} = t_3^2 - 2t_2^2 + t_1^2 + 2t_0(2t_2 - t_3 - t_1),\tag{25}$$

or reordered:

$$2t_0(-t_1 + 2t_2 - t_3) = \frac{2h^2}{v^2} - (t_3^2 - 2t_2^2 + t_1^2). \quad (26)$$

We define the travelttime between two adjacent receivers to be $t_h = h / v$, giving:

$$2t_0(-t_1 + 2t_2 - t_3) = 2t_h^2 - t_3^2 + 2t_2^2 - t_1^2, \quad (27)$$

then finally a compact form for t_0 follows as:

$$t_0 = \frac{t_1^2 - 2t_2^2 + t_3^2 - 2t_h^2}{2(t_1 - 2t_2 + t_3)}. \quad (28)$$

To obtain z_0 , solve the first line of equation (22):

$$-2hz_0 + h^2 + 2z_1h = v^2(t_2^2 - 2t_2t_0 - t_1^2 + 2t_1t_0), \quad (29)$$

we get:

$$2hz_0 = v^2(-t_2^2 + 2t_2t_0 + t_1^2 - 2t_1t_0) + h^2 + 2z_1h, \quad (30)$$

or

$$z_0 = \frac{1}{2h} \left[2t_0v^2(t_2 - t_1) + v^2(t_1^2 - t_2^2) + h^2 + 2z_1h \right]. \quad (31)$$

We then get x_0 from the first line of equation (20)

$$x_0^2 + (z_1 - z_0)^2 = v^2(t_1 - t_0)^2, \quad (32)$$

or

$$x_0^2 = v^2(t_1 - t_0)^2 - (z_1 - z_0)^2, \quad (33)$$

then

$$x_0 = \pm \sqrt{v^2(t_1 - t_0)^2 - (z_1 - z_0)^2}. \quad (34)$$

Since we are using cylindrical coordinates the value of r_0 is equivalent to the positive value of x_0 , or:

$$r_0 = \text{sqrt} \left(v^2(t_1 - t_0)^2 - (z_0 - z_1)^2 \right). \quad (35)$$