Inverting absorptive reflections: an inverse series tutorial

Kris Innanen

ABSTRACT

Non-linear inverse scattering, when it is applied to the big interesting seismic imaging and inversion problems, can be very complicated, and in a tutorial setting one risks losing all the basic concepts and insights in a morass of arithmetic. We can usefully proceed, however, by considering direct inversion within a highly simplified model: reflections from a single interface at a known depth, with known medium parameters above the interface and unknown medium parameters below. The seismic data reduce in this case to a single reflection coefficient, and the medium to be solved for reduces to a few scalar values. We consider the case of an absorptive target medium. A simple absorptive reflection coefficient may be expanded about small parameter contrasts and incidence angles, and used, angle by angle (AVA), or frequency by frequency (AVF), to directly determine simultaneous wavespeed and Q contrasts. Linear and non-linear inversion may occur algebraically or using an inverse series. The latter is evidently the better approach of the two, with the former becoming less tractable for cases involving large angles and large contrasts.

INTRODUCTION

A main aim here is to provide a sense of how inverse solutions that directly transform seismic data information into estimates of medium properties can be derived. We will do this within the framework of AVA inversion, a reduction of the full problem of seismic inversion to the determination of the parameter contrasts that have given rise to a particular reflection coefficient.

Whereas Zhang and Weglein (2009a,b) have provided formulas for direct non-linear inversion of acoustic and elastic targets with seismic reflection coefficients as input, similar formulas for absorptive reflection data are to date unavailable in the literature. We will treat this case. The reflection coefficient associated with a plane contrast in an-elastic or an-acoustic medium parameters (e.g., White, 1965; Borcherdt, 1977; Kjartansson, 1979; Krebes, 1984; Lam et al., 2004; de Hoop et al., 2005; Lines et al., 2008) contains characteristic variations which could in principle be used to determine the relative changes in each parameter at the point of contrast. Direct inverse scattering procedures, in the limit as a set of absorptive volume scatterers combine to produce a specular reflector, in fact are seen to make exclusive use of this kind of amplitude information (Innanen and Weglein, 2007; Innanen and Lira, 2009). The variations in question are relatively small, and the literature is divided on the ease of their detectability. Over a decade ago, absorption-specific reflection coefficient variations were reported to be un-observable as AVO/AVA behavior, when synthetic viscoelastic data were examined at a single frequency over a suite of incidence angles (Samec and Blangy, 1992). However, more recently, field measurements of reflection coefficients associated with certain low-Q fluid filled reservoirs have been described as “strongly frequency dependent” (Odebeatu et al., 2006). The angle- and frequency-dependence of dispersive reflection coefficients are coupled, hence we would seem to be missing a clear and consistent picture. Still, the more recent investigations (Chapman et al., 2006; Ren et al., 2009) all come down at least implicitly on the "is detectable" side, suggesting a trend in that direction, and indeed the earlier inconsistencies are plausibly explainable as being due to particular choices of viscoelastic model, medium contrasts, etc., rather than more fundamental issues.

Let us proceed assuming that variations in the reflection coefficient due to absorption and dispersion of a target reflector have been unambiguously measured*, and consider the problem of separately estimating the influence of medium parameters (including Q) on them. We will demonstrate that a simple, two-parameter dispersive reflection coefficient may be expressed in terms of a variety of different plane-wave variables, expanded about small parameter contrasts and incidence angles, and used to directly determine the target's properties.

ABSORPTIVE REFLECTION COEFFICIENTS

We will work in two spatial dimensions. Consider a wavefield P_0, propagating in a homogeneous, source-free, non-absorptive medium (medium 0) according to

$$\left[\nabla^2 + \frac{\omega^2}{c_0^2}\right] P_0(x, z, \omega) = 0, \tag{1}$$

and consider P_1 propagating in an absorptive medium (medium 1) according to

$$\left[\nabla^2 + \frac{\omega^2}{c_1^2}\left(1 + \frac{F(\omega)}{Q_1}\right)^2\right] P_1(x, z, \omega) = 0, \tag{2}$$

where (Aki and Richards, 2002)

$$F(\omega) = \frac{i}{2} - \frac{1}{\pi}\log\left(\frac{\omega}{\omega_r}\right), \tag{3}$$

and ω_r is an arbitrary reference frequency. Fourier transforming equations (1) and (2), we have

$$k_x^2 + k_z^2 = \frac{\omega^2}{c_0^2}, \tag{4}$$

and

$$k_x^2 + k_z'^2 = \frac{\omega^2}{c_1^2}\left(1 + \frac{F(\omega)}{Q_1}\right)^2, \tag{5}$$

the first of which implies a range of possible relationships based on plane wave geometry, e.g.:

$$\begin{aligned} k_x &= \frac{\omega}{c_0}\sin\theta, \\ k_z &= \frac{\omega}{c_0}\cos\theta, \end{aligned} \tag{6}$$

*And, indeed, that these variations have been correctly ascribed to absorptive reflectivity, and not to, say, thin layering.

where θ represents the plane wave angle measured away from the direction of positive z. If P_0 is incident upon a plane boundary separating medium 0 from medium 1, continuity of the field (e.g., pressure) across the interface requires that there be a reflection coefficient

$$R = \frac{k_z - k_z'}{k_z + k_z'}, \tag{7}$$

which, because of equations (3)–(5), we anticipate to be complex, frequency-dependent, and expressive of the an-acoustic properties of the target medium in some hopefully useful ways.

There is some room for choice in studying the angle-dependence of this coefficient in terms of the quantities k_x, k_z, and ω. For instance, if frequency ω is a parameter,

$$R_\omega(\theta) = \frac{c_1\cos\theta - c_0\left[1+Q_1^{-1}F(\omega)\right]\sqrt{1-\frac{c_1^2}{c_0^2}\left[1+Q_1^{-1}F(\omega)\right]^{-2}\sin^2\theta}}{c_1\cos\theta + c_0\left[1+Q_1^{-1}F(\omega)\right]\sqrt{1-\frac{c_1^2}{c_0^2}\left[1+Q_1^{-1}F(\omega)\right]^{-2}\sin^2\theta}}, \tag{8}$$

whereas, if k_x is a parameter,

$$R_{kx}(\theta) = \frac{c_1\cos\theta - c_0\left[1+Q_1^{-1}F_{kx}(\theta)\right]\sqrt{1-\frac{c_1^2}{c_0^2}\left[1+Q_1^{-1}F_{kx}(\theta)\right]^{-2}\sin^2\theta}}{c_1\cos\theta + c_0\left[1+Q_1^{-1}F_{kx}(\theta)\right]\sqrt{1-\frac{c_1^2}{c_0^2}\left[1+Q_1^{-1}F_{kx}(\theta)\right]^{-2}\sin^2\theta}}, \tag{9}$$

where

$$F_{kx}(\theta) = \frac{i}{2} - \frac{1}{\pi}\log\left(\frac{k_x c_0}{\omega_r \sin\theta}\right), \tag{10}$$

and if k_z is a parameter,

$$R_{kz}(\theta) = \frac{c_1\cos\theta - c_0\left[1+Q_1^{-1}F_{kz}(\theta)\right]\sqrt{1-\frac{c_1^2}{c_0^2}\left[1+Q_1^{-1}F_{kz}(\theta)\right]^{-2}\sin^2\theta}}{c_1\cos\theta + c_0\left[1+Q_1^{-1}F_{kz}(\theta)\right]\sqrt{1-\frac{c_1^2}{c_0^2}\left[1+Q_1^{-1}F_{kz}(\theta)\right]^{-2}\sin^2\theta}}, \tag{11}$$

where

$$F_{kz}(\theta) = \frac{i}{2} - \frac{1}{\pi}\log\left(\frac{k_z c_0}{\omega_r \cos\theta}\right). \tag{12}$$

SERIES EXPANSIONS OF R

Examples of quantities embedded in R that are, from a geophysical point of view, sometimes small, but not always, are angle of incidence, the relative change in wavespeed (from c_0 to c_1), and the relative change in Q^{-1} (from 0 to Q_1^{-1}). As measures of the latter two quantities we define

$$\begin{aligned} \alpha &= 1 - \frac{c_0^2}{c_1^2}, \\ \beta &= \frac{1}{Q_1}, \end{aligned} \tag{13}$$

and as a measure of the first, we consider R as a function of $\sin^2\theta$. Depending on the parametrization, the presence of dispersion can bring an additional level of complexity in the θ dependence beyond the acoustic case. If ω is a parameter R expands as

$$\begin{aligned}R_\omega(\theta) = &\left[\left(\frac{1}{4}\alpha - \frac{1}{2}F(\omega)\beta\right) + \left(\frac{1}{8}\alpha^2 + \frac{1}{4}F^2(\omega)\beta^2\right) + ...\right]\left(\sin^2\theta\right)^0\\ &+ \left[\left(\frac{1}{4}\alpha - \frac{1}{2}F(\omega)\beta\right) + \left(\frac{1}{4}\alpha^2 - \frac{1}{2}F(\omega)\alpha\beta + \frac{3}{4}F^2(\omega)\beta^2\right) + ...\right]\left(\sin^2\theta\right)^1\\ &+ \left[\left(\frac{1}{4}\alpha - \frac{1}{2}F(\omega)\beta\right) + \left(\frac{5}{16}\alpha^2 - \frac{3}{4}F(\omega)\alpha\beta + F^2(\omega)\beta^2\right) + ...\right]\left(\sin^2\theta\right)^2\\ &+ ... \;.\end{aligned} \tag{14}$$

If k_z is a parameter, R instead has the form

$$\begin{aligned}\tilde{R}_{kz}(\theta) = &\left\{\left[\frac{1}{4}\alpha - \frac{1}{2}\tilde{F}_{kz}\beta\right] + \left[\frac{1}{8}\alpha^2 + \frac{1}{4}\tilde{F}_{kz}^2\beta^2\right] + ...\right\}\left(\sin^2\theta\right)^0\\ &+ \left\{\left[\frac{1}{4}\alpha - \left(\frac{1}{2}\tilde{F}_{kz} - \frac{1}{2\pi}\right)\beta\right] + \left[\frac{1}{4}\alpha^2 - \frac{1}{2}\tilde{F}_{kz}\alpha\beta + \left(\frac{3}{4}\tilde{F}_{kz}^2 - \frac{1}{4\pi}\right)\beta^2\right] + ...\right\}\left(\sin^2\theta\right)^1\\ &+ \left\{\left[\frac{1}{4}\alpha - \left(\frac{1}{2}\tilde{F}_{kz} - \frac{3}{4\pi}\right)\beta\right] + \left[\frac{5}{16}\alpha^2 - \left(\frac{3}{4}\tilde{F}_{kz} - \frac{1}{4\pi}\right)\alpha\beta\right.\right.\\ &\left.\left. + \left(\tilde{F}_{kz}^2 - \frac{7}{8\pi}\tilde{F}_{kz} + \frac{1}{16\pi^2}\right)\beta^2\right] + ...\right\}\left(\sin^2\theta\right)^2\\ &+ ...,\end{aligned} \tag{15}$$

where

$$\tilde{F}_{kz} = \frac{i}{2} - \frac{1}{\pi}\log\left(\frac{k_z c_0}{\omega_r}\right). \tag{16}$$

Alternatively, if R_{kz} is expanded only in the angle variations that occur absent dispersion, we have instead

$$\begin{aligned}R_{kz}(\theta) = &\left[\left(\frac{1}{4}\alpha - \frac{1}{2}F_{kz}(\theta)\beta\right) + \left(\frac{1}{8}\alpha^2 + \frac{1}{4}F_{kz}^2(\theta)\beta^2\right) + ...\right]\left(\sin^2\theta\right)^0\\ &+ \left[\left(\frac{1}{4}\alpha - \frac{1}{2}F_{kz}(\theta)\beta\right) + \left(\frac{1}{4}\alpha^2 - \frac{1}{2}F_{kz}(\theta)\alpha\beta + \frac{3}{4}F_{kz}^2(\theta)\beta^2\right) + ...\right]\left(\sin^2\theta\right)^1\\ &+ \left[\left(\frac{1}{4}\alpha - \frac{1}{2}F_{kz}(\theta)\beta\right) + \left(\frac{5}{16}\alpha^2 - \frac{3}{4}F_{kz}(\theta)\alpha\beta + F_{kz}^2(\theta)\beta^2\right) + ...\right]\left(\sin^2\theta\right)^2\\ &+ ... \;.\end{aligned} \tag{17}$$

THE FORWARD PROBLEM AND THE INVERSE PROBLEM

The forward problem we consider in this paper is the calculation of R over a range of the variables k_x, ω (i.e., the Fourier conjugates of offset and time respectively, which can then be transformed to k_z, θ as desired), given α and β. The inverse problem is the exact or approximate determination of α and β from values of R over a range of one of these variables. In the abstract the inversion is referred to as AVA or AVF, to correspond to the use of angle or frequency respectively as this variable. Any time in the following examples there are two parameters to be solved for, at least two data will be required, and they will be produced by varying the values of either frequency or angle and assuming that R is available at these values. In any practical implementation of these formulas we would naturally use as many data as we had, and fit the parameters with an appropriate regression. Here for the sake of simplicity we will assume exact data, in which case, since there are at most two parameters to solve for, we will use at most two data.

INVERSION OF R AT NORMAL INCIDENCE FOR A SINGLE PARAMETER CONTRAST

We first consider the problem of a wave field impinging at normal incidence on a plane contrast in Q by setting $\theta = 0$ and $\alpha = 0$. Choosing ω as a parameter, i.e., beginning with equation (14), we then have $R_\omega = R_\omega(\theta)|_{\theta=0}$:

$$\begin{aligned} R_\omega &= -\frac{1}{2}F(\omega)\beta + \frac{1}{4}F^2(\omega)\beta^2 - ... \\ &= -\frac{\frac{1}{2}F(\omega)\beta}{1+\frac{1}{2}F(\omega)\beta}. \end{aligned} \tag{18}$$

Algebraic inverse solution

The inverse problem can be solved exactly given R_ω at any one fixed frequency by isolating β in equation (18), and that solution, if desired, can itself be expressed as a series:

$$\begin{aligned} \beta &= -\frac{2}{F(\omega)}\frac{R_\omega}{1+R_\omega} \\ &= -\frac{2}{F(\omega)}\left(R_\omega - R_\omega^2 + ...\right). \end{aligned} \tag{19}$$

If β is small, we may be willing to make the first order approximation

$$\beta \approx -\frac{2}{F(\omega)}R_\omega, \tag{20}$$

but in equation (19) we have a formula for β accurate to any desired order in the data R_ω.

Inverse series

It is only rarely possible to solve exactly and non-linearly for a medium parameter in terms of the reflection coefficient through simple algebra, as above, and the occasions when it is will tend to be too simple to be interesting. An alternative approach that continues to work as the expressions for the data become increasingly complicated, instead involves forming an inverse series for the desired quantity:

$$\beta = \beta_1 + \beta_2 + ..., \tag{21}$$

where β_n is defined to be the component of β that is n'th order in R_ω, substituting it into equation (18):

$$R_\omega = -\frac{1}{2}F(\omega)\beta_1 - \frac{1}{2}F(\omega)\beta_2 + \frac{1}{4}F^2(\omega)\beta_1^2 + ..., \tag{22}$$

and equating like orders, from which we determine

$$\beta_1 = -\frac{2}{F(\omega)}R_\omega, \tag{23}$$

at first order,

$$\beta_2 = \frac{1}{2}F(\omega)\beta_1^2 = \frac{2}{F(\omega)}R_\omega^2, \tag{24}$$

at second order, etc. The same inverse formula determined algebraically in equation (19) is recovered. At this level of complexity (one parameter, normal incidence) it is not clear that either approach is preferable, but that will change.

INVERSION OF R AT NORMAL INCIDENCE FOR A TWO PARAMETER CONTRAST

If again a reflection coefficient (expressed with ω as a parameter) due to a normally incident field was measured, but this time with $\alpha \neq 0$ and $\beta \neq 0$, i.e., having the form

$$R_\omega = \left(\frac{1}{4}\alpha - \frac{1}{2}F(\omega)\beta\right) + \left(\frac{1}{8}\alpha^2 + \frac{1}{4}F^2(\omega)\beta^2\right) + ..., \tag{25}$$

the wrinkle for the inverse problem is that simultaneous variations in α and β now account for the amplitude content of R_ω.

Algebraic inverse solution

Truncating equation (25) at first order:

$$R_\omega \approx \frac{1}{4}\alpha - \frac{1}{2}F(\omega)\beta, \tag{26}$$

we notice that within this approximation the frequency dependence of R_ω is due solely to variations in β. Hence if we take R at two distinct frequencies, R_{ω_1} and R_{ω_2}, and subtract them, by equation (26) we have isolated β:

$$\beta \approx 2\frac{R_{\omega_1} - R_{\omega_2}}{F(\omega_2) - F(\omega_1)}. \tag{27}$$

Likewise we may separate out the R behavior due to α:

$$\alpha \approx 4\frac{R_{\omega_1}F(\omega_2) - R_{\omega_2}F(\omega_1)}{F(\omega_2) - F(\omega_1)}. \tag{28}$$

Because of the fortuitous lack of an $\alpha\beta$ cross-term at second order in equation (25), this scheme also works if we truncate the reflection coefficient series in equation (25) at second order:

$$R_\omega \approx \frac{1}{4}\alpha - \frac{1}{2}F(\omega)\beta + \frac{1}{8}\alpha^2 + \frac{1}{4}F^2(\omega)\beta^2. \tag{29}$$

Again selecting two frequencies and extracting from the data R_{ω_1} and R_{ω_2}, we have, to replace the first-order inverse formula in equation (27), the second-order inverse formula

$$\beta \approx \frac{[F(\omega_1) - F(\omega_2)] - \sqrt{[F(\omega_1) - F(\omega_2)]^2 - 4[F^2(\omega_1) - F^2(\omega_2)](R_{\omega_2} - R_{\omega_1})}}{[F^2(\omega_1) - F^2(\omega_2)]}; \tag{30}$$

with this value of β in hand, from equation (29) we may then solve for α, replacing the first order inverse formula in equation (28) with the second-order inverse formula:

$$\alpha \approx -1 + \sqrt{1 + 8\left(R_\omega + \frac{1}{2}F(\omega)\beta - \frac{1}{4}F^2(\omega)\beta^2\right)}, \tag{31}$$

in principle using any desired frequency component R_ω and its corresponding $F(\omega)$.

Inverse series

Answers with the same degree of accuracy can be produced by again forming inverse series $\alpha = \alpha_1 + \alpha_2 + ...$ and $\beta = \beta_1 + \beta_2 + ...$, substituting them into equation (25):

$$R_\omega = \frac{1}{4}\alpha_1 + \frac{1}{4}\alpha_2 - \frac{1}{2}F(\omega)\beta_1 - \frac{1}{2}F(\omega)\beta_2 + \frac{1}{8}\alpha_1^2 + \frac{1}{4}F^2(\omega)\beta_1^2 + ..., \tag{32}$$

and equating like orders. Truncating the inverse series at first order, we obtain

$$\begin{aligned} \alpha \approx \alpha_1 &= 4\frac{R_{\omega_1}F(\omega_2) - R_{\omega_2}F(\omega_1)}{F(\omega_2) - F(\omega_1)}, \\ \beta \approx \beta_1 &= 2\frac{R_{\omega_1} - R_{\omega_2}}{F(\omega_2) - F(\omega_1)}, \end{aligned} \tag{33}$$

i.e., the same result as was derived algebraically in equations (27) and (28). Truncating at second order, we instead obtain

$$\begin{aligned} \alpha \approx \alpha_1 + \alpha_2 &= 4\frac{\left(R_{\omega_1} + R_{\omega_1}^{(2)}\right)F(\omega_2) - \left(R_{\omega_2} + R_{\omega_2}^{(2)}\right)F(\omega_1)}{F(\omega_2) - F(\omega_1)}, \\ \beta \approx \beta_1 + \beta_2 &= 2\frac{\left(R_{\omega_1} + R_{\omega_1}^{(2)}\right) - \left(R_{\omega_2} + R_{\omega_2}^{(2)}\right)}{F(\omega_2) - F(\omega_1)}, \end{aligned} \tag{34}$$

where

$$\begin{aligned} R_{\omega_1}^{(2)} &\equiv -\frac{1}{8}\alpha_1^2 - \frac{1}{4}F^2(\omega_1)\beta_1^2, \\ R_{\omega_2}^{(2)} &\equiv -\frac{1}{8}\alpha_1^2 - \frac{1}{4}F^2(\omega_2)\beta_1^2, \end{aligned} \tag{35}$$

or, in more compact form:

$$\alpha \approx \alpha_1 - \frac{1}{2}\alpha_1^2 - \beta_1^2\left(\frac{F^2(\omega_1)F(\omega_2) - F^2(\omega_2)F(\omega_1)}{F(\omega_2) - F(\omega_1)}\right),$$
$$\beta \approx \beta_1 + \frac{1}{2}[F(\omega_2) + F(\omega_1)]\beta_1^2. \tag{36}$$

INVERSION OF R AT OBLIQUE INCIDENCE FOR A TWO PARAMETER CONTRAST

So far we have made explicit use of the frequency dependence of R to separately determine c and Q; we therefore refer to what we have done as linear and/or non-linear AVF inversion. As we will see next, variations in α and β may alternatively be determined by using the angle dependence of the reflection coefficient, making implicit rather than explicit use of its frequency dependence. This then would be a form of AVA inversion, tuned to the particular problem of absorption. For those familiar with the work of Dasgupta and Clark (1998), this may sound like old news, but we emphasize that their QVO method is based on attenuation along increasing path-lengths of a wave in an absorptive medium *above* a reflector. Here, we use the reflection coefficient, that is, we make use of information coming from the reflector itself: in point of fact, there *is* no Q in the layer overlying our reflector.

Algebraic inversion

Choosing k_z as a parameter, we truncate the series for R in equation (17) at first order in α, β, and $\sin^2\theta$:

$$R_{kz}(\theta) \approx \left(\frac{1}{4}\alpha - \frac{1}{2}F_{kz}(\theta)\beta\right)\left(1+\sin^2\theta\right). \tag{37}$$

Dividing through by $1+\sin^2\theta$, we have, still to first order in $\sin^2\theta$,

$$R_{kz}(\theta)\cos^2\theta \approx \frac{1}{4}\alpha - \frac{1}{2}F_{kz}(\theta)\beta, \tag{38}$$

so α and β may be separated by choosing two angles of incidence θ_1 and θ_2:

$$\beta \approx 2\frac{R_{kz}(\theta_1)\cos^2\theta_1 - R_{kz}(\theta_2)\cos^2\theta_2}{F_{kz}(\theta_2) - F_{kz}(\theta_1)}, \tag{39}$$

and

$$\alpha \approx 4\frac{R_{kz}(\theta_1)\cos^2\theta_1 F_{kz}(\theta_2) - R_{kz}(\theta_2)\cos^2\theta_2 F_{kz}(\theta_1)}{F_{kz}(\theta_2) - F_{kz}(\theta_1)}. \tag{40}$$

Because $R_{kz}(\theta)$ has a non-zero term in $\alpha\beta$ at first order in $\sin^2\theta$, the algebraic determination of higher order corrections for α and β using the angle dependence of R, while probably not impossible, will likely try the interested researcher's patience.

Inverse series

In contrast, it is reasonably straightforward to construct non-linear formulas for α and β using the inverse series approach. Again forming $\alpha = \alpha_1 + \alpha_2 + ...$ and $\beta = \beta_1 + \beta_2 + ...$, substituting them into equation (17) and equating like orders, we recover a similar expression for the linearly determined α_1 and β_1:

$$\beta_1 = 2\frac{R_{kz}(\theta_1)\cos^2\theta_1 - R_{kz}(\theta_2)\cos^2\theta_2}{F_{kz}(\theta_2) - F_{kz}(\theta_1)}, \tag{41}$$

and

$$\alpha_1 = 4\frac{R_{kz}(\theta_1)\cos^2\theta_1 F_{kz}(\theta_2) - R_{kz}(\theta_2)\cos^2\theta_2 F_{kz}(\theta_1)}{F_{kz}(\theta_2) - F_{kz}(\theta_1)}. \tag{42}$$

However, we may continue straightforwardly to higher order. For instance, at second order we have

$$\begin{aligned}
\alpha \approx \alpha_1 + \alpha_2 &= 4\frac{[R_{kz}(\theta_1) + \tilde{R}_{kz}(\theta_1)]F_{kz}(\theta_2)\cos^2\theta_1 - [R_{kz}(\theta_2) + \tilde{R}_{kz}(\theta_2)]F_{kz}(\theta_1)\cos^2\theta_2}{F_{kz}(\theta_2) - F_{kz}(\theta_1)}, \\
\beta \approx \beta_1 + \beta_2 &= 2\frac{[R_{kz}(\theta_1) + \tilde{R}_{kz}(\theta_1)]\cos^2\theta_1 - [R_{kz}(\theta_2) + \tilde{R}_{kz}(\theta_2)]\cos^2\theta_2}{F_{kz}(\theta_2) - F_{kz}(\theta_1)}
\end{aligned} \tag{43}$$

where

$$\tilde{R}_{kz}(\theta) = -\frac{1}{\cos^2\theta}\left(\frac{1}{4}\alpha_1^2 + \frac{1}{2}F_{kz}^2(\theta)\beta_1^2\right) - \left(\frac{1}{2}\alpha_1 + F_{kz}(\theta)\beta_1\right)^2 \sin^2\theta. \tag{44}$$

NUMERIC EXAMPLE

To provide some numerical illustration of these ideas, we consider the normal incidence two parameter case, i.e., we implement the formulas derived in equations (33)–(36). The real part of the reflection coefficient R associated with an acoustic medium (c_0 =1500m/s) overlying an absorptive medium (c_1 =1800m/s, Q_1 =10) is illustrated in Figure 1, with, for comparison, the associated acoustic reflection coefficient (i.e., with $1/Q_1$ =0). The variation with frequency of the reflection coefficient permits the two parameters to be independently determined.

In Figure 2 we illustrate the use of these formulas to invert for c_1 and Q_1. Target medium properties are determined using two values of the absorptive R depicted in Figure 1, one fixed at 1 Hz, the other taking on values ranging from 2-100 Hz. Roughly 100 estimates are calculated, and plotted as a function of this second frequency value. In Figure 2a the target Q value (in gold) is recovered to first order using equation (33). This is in effect the inverse Born approximation of Q_1. In Figure 2b, the target Q value is recovered to second order (i.e., one order “beyond Born”) using equation (36). In Figure 2c, the target wavespeed value (in blue) is recovered to first order using equation (33). And, in Figure

2d, the target wavespeed value is recovered, accurate to second order, using equation (36). In addition to the bulk tendency of the inverted properties to converge towards their exact values, with a significant increase in accuracy from first to second order, it is worth noting that the spurious variation of the inversion result with experimental variables (in this case frequency), a characteristic of the inverse Born approximation for large contrast targets, diminishes as order increases.

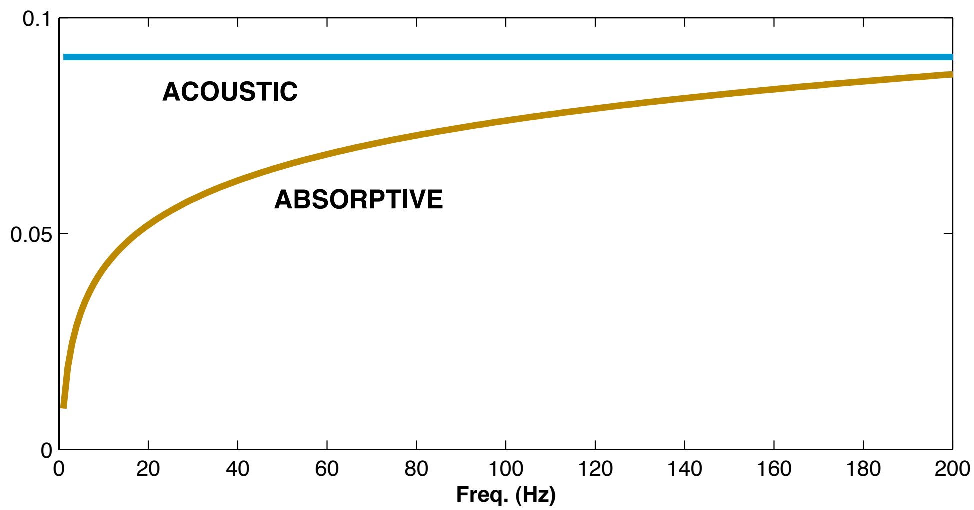

FIG. 1. Reflection coefficients as a function of frequency.

CONCLUSIONS

Beyond demonstrating how inverse series solutions are found, in concluding we might emphasize some benefits of this style of inversion. We have seen that up to certain levels of mathematical complexity†, the direct solution for wavespeed c and quality factor Q contrasts in terms of the reflection coefficient can be carried out algebraically or with a series; but, at some point, the former becomes unwieldy, and ultimately intractable. The power of inverse scattering series methods is that we may keep piling complexity upon complexity, including propagation effects, transmission losses, multidimensionality of the perturbation, and in principle any number of medium parameters, and the approach (defining an inverse series to be substituted into the forward problem and solved for order by order) *keeps working*.

The inverse series results and their derivations in this paper, in addition to providing new and potentially useful formulas for inversion, are meant to act as a conceptual introduction to inverse *scattering* series methods. So, we also emphasize that the former (with a few exceptions) represent a special case of the latter, and capture much of their flavour. The formula in equation (44), for instance, is reproduced exactly by linearizing the full absorptive inverse scattering problem with the constraint that the solution has to be a Heaviside

†Complex, meaning, for instance, large versus small number of parameters undergoing contrasts, high versus low order of non-linearity, or large versus small angles of incidence

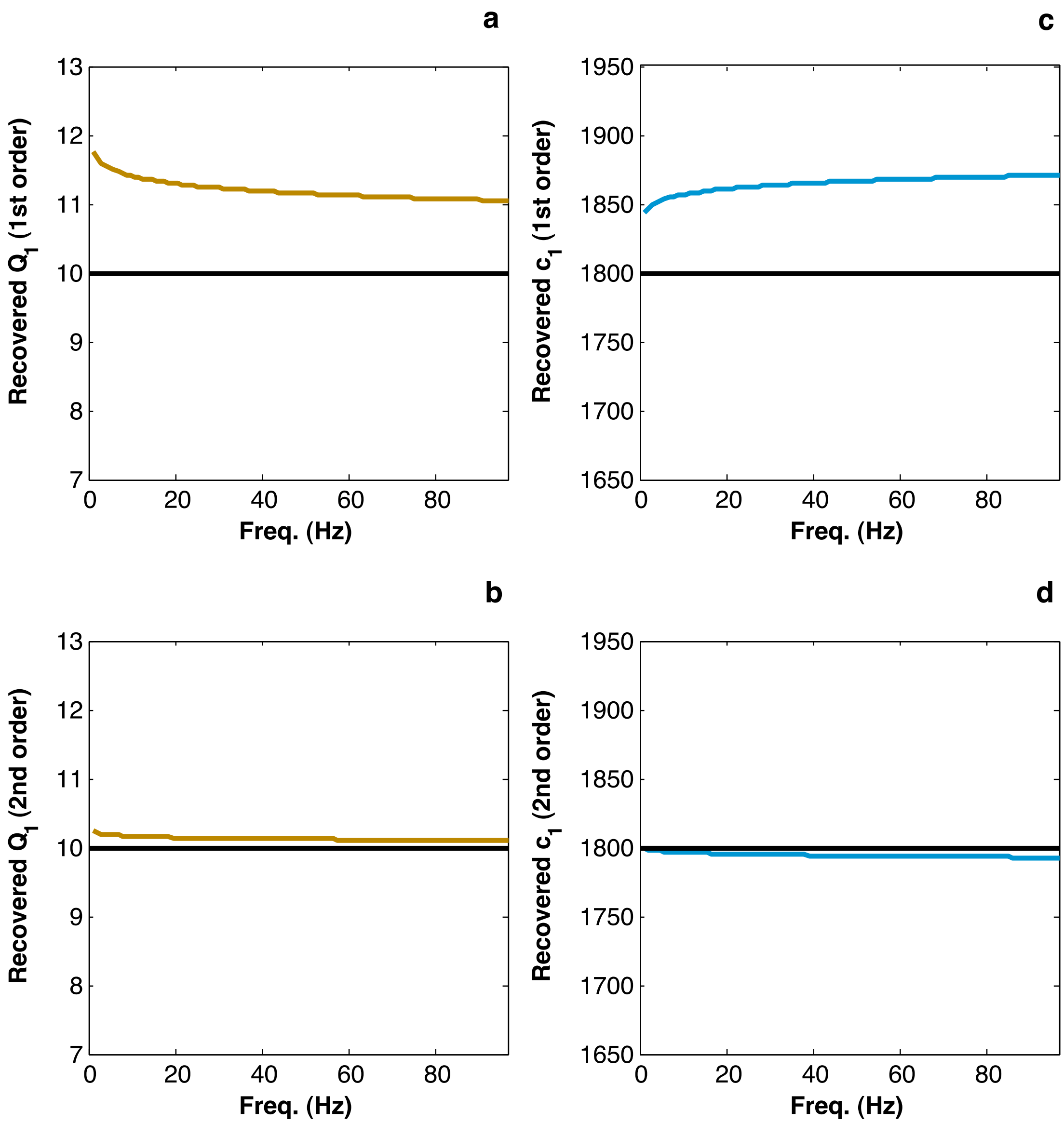

FIG. 2. Target medium properties recovered over a range of frequency pairs: (a) Q_1, to first order, (b) Q_1, to second order, (c) c_1, to first order, (d) c_1, to second order. Exact property values are displayed in black.

function (Innanen and Weglein, 2007, equation 57).

In spite of the fact that the inverse scattering series has been around for a long time (Moses, 1956; Razavy, 1975; Weglein et al., 1981; Stolt and Jacobs, 1980; Weglein et al., 2003), the approach taken in this paper does have some new and unusual features beyond its application to absorptive problems. In fact, it is an example of a recently-suggested alteration of the standard program of inverse scattering. Here, we alter the forward scattering problem to approximate either a simple experiment, or only a certain type of event, before any inverse steps are taken (Innanen, 2008); then the altered forward series is inverted, with perturbations being reconstructed order-by-order in what are now very differently defined data. Although the final formulas and algorithms that arise can all be derived by analyzing and breaking up the full inverse scattering series, it may be that confining all manipulations and approximations to the forward part of the problem is advantageous, since the forward scattering problem is more straightforwardly linked to wave physics than its inverse counterpart.

REFERENCES

Aki, K., and Richards, P. G., 2002, Quantitative Seismology: University Science Books, 2nd edn.

Borcherdt, R. D., 1977, Reflection and refraction of type-II S waves in attenuating media: Bull. Seism. Soc. Am., **67**, 43–67.

Chapman, M., Liu, E., and Li, X. Y., 2006, The influence of fluid-sensitive dispersion and attenuation on AVO analysis: Geophysical Journal International, **167**, No. 1, 89–105.

Dasgupta, R., and Clark, R., 1998, Estimation of Q from surface seismic reflection data: Geophysics, **63**, 2120–2128.

de Hoop, A. T., Lam, C. H., and Kooij, B. J., 2005, Parametrization of acoustic boundary absorption and dispersion properties in time domain source/receiver reflection measurement: J. Acoust. Soc. Am., **118**, 654–660.

Innanen, K. A., 2008, A direct non-linear inversion of primary wave data reflecting from extended, heterogeneous media: Inverse Problems, **24**, 035,021.

Innanen, K. A., and Lira, J. E., 2009, Direct non-linear Q compensation of seismic primaries reflecting from a stratified, 2-parameter absorptive medium: Geophysics, *in press*.

Innanen, K. A., and Weglein, A. B., 2007, On the construction of an absorptive-dispersive medium model via direct linear inversion of reflected seismic primaries: Inverse Problems, **23**, 2289–2310.

Kjartansson, E., 1979, Attenuation of seismic waves in rocks and applications in energy exploration: Ph.D. thesis, Stanford University.

Krebes, E. S., 1984, On the reflection and transmission of viscoelastic waves–some numerical results: Geophysics, **49**, 1374–1380.

Lam, C. H., Kooij, B. J., and de Hoop, A. T., 2004, Impulsive sound reflection from an absorptive and dispersive planar boundary: J. Acoust. Soc. Am., **116**, 677–685.

Lines, L. R., Vasheghani, F., and Treitel, S., 2008, Reflections on Q: CSEG Recorder, **Dec**, 36–38.

Moses, H., 1956, Calculation of scattering potential from reflection coefficients: Phys. Rev., , No. 102, 559–567.

Odebeatu, E., Zhang, J., Chapman, M., Liu, E., and Li, X. Y., 2006, Application of spectral decomposition to detection of dispersion anomalies associated with gas saturation: The Leading Edge, **2**, 206–210.

Razavy, M., 1975, Determination of the wave velocity in an inhomogeneous medium from reflection data: J. Acoust. Soc. Am., **58**, 956–963.

Ren, H., Goloshubin, G., and Hilterman, F. J., 2009, Poroelastic analysis of amplitude-versus-frequency variations: Geophysics, **74**, No. 6, N49–N54.

Samec, P., and Blangy, J. P., 1992, Viscoelastic attenuation, anisotropy, and AVO: Geophysics, **57**, 441.

Stolt, R. H., and Jacobs, B., 1980, Inversion of seismic data in a laterally heterogeneous medium: Stanford Exploration Project, **24**.

Weglein, A. B., Araújo, F. V., Carvalho, P. M., Stolt, R. H., Matson, K. H., Coates, R. T., Corrigan, D., Foster, D. J., Shaw, S. A., and Zhang, H., 2003, Inverse scattering series and seismic exploration: Inverse Problems, , No. 19, R27–R83.

Weglein, A. B., Boyce, W. E., and Anderson, J. E., 1981, Obtaining three-dimensional velocity information directly from reflection seismic data: An inverse scattering formalism: Geophysics, **46**, No. 8, 1116–1120.

White, J. E., 1965, Reflections from lossy media: Journal of the Acoustical Society of America, **38**, 604–607.

Zhang, H., and Weglein, A. B., 2009a, Direct non-linear inversion of multi-parameter 1D acoustic media using the inverse scattering series: Geophysics, *in press*.

Zhang, H., and Weglein, A. B., 2009b, Direct non-linear inversion of multi-parameter 1D elastic media using the inverse scattering series: Geophysics, *in press*.