Using three-point difference approximation to improve absorbing boundary conditions for elastic wave modelling

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ABSTRACT

Absorbing boundary conditions are partial difference equations, for which only forward or backward difference approximations instead of central difference approximations can be used. Usually two-point approximations are used for first order derivatives; however, three-point approximations result in less reflection from the computational boundaries. A comparison of the two-point and three-point approximations is illustrated.

INTRODUCTION

Computational boundary condition problems have been a persistent topic in the field of wave modelling and migration. The most cited method about boundary conditions is the "absorbing boundary conditions" proposed by Clayton and Engquist in 1977. The proposed absorbing boundary conditions A1, A2, and A3 are, respectively, first, second, and third order partial differential equations. Among these boundary conditions, the simplest and thus most popular one is A1. For A1, we can use the two-point forward or backward difference approximations, as suggested. However, when we use the three-point approximation, which is more accurate, the boundary reflections are reduced more.

FIRST ORDER DERIVATIVE APPROXIMATIONS

First order derivatives can be approximated by a two-point forward or backward difference approximation. The two-point backward approximation can be written as

$$f'(x_i) = \frac{f(x_i) - f(x_{i-1})}{h}. \tag{1}$$

The two-point forward and backward difference approximations have truncation errors of the order of $O(h)$.

First order derivatives can also be approximated by a three-point forward or backward difference approximation. The three-point forward backward approximation can be written as

$$f'(x_i) = \frac{f(x_{i-2}) - 4f(x_{i-1}) + 3f(x_i)}{2h}. \tag{2}$$

The three-point forward and backward difference approximations have truncation errors of the order of $O(h^2)$.

ABSORBING BOUNDARY CONDITIONS

The absorbing boundary conditions A1 (Clayton and Engquist, 1977) for the bottom boundary of a 2D elastic subsurface model can be written as

$$\begin{aligned} U_z + U_t/\beta = 0 \\ V_z + V_t/\alpha = 0 \end{aligned}, \tag{3}$$

where U and V are, respectively, the vertical and horizontal particle velocity; α and β are, respectively, P-wave and S-wave velocity.

As suggested by Clayton and Engquist (1977), we can use two-point backward difference operators with respect to z and t. Thus, system (3) in the Madariaga-Virieux staggered-grid scheme (Virieux, 1986) can be written as

$$\begin{aligned} \frac{U_{i,j}^{k-1/2} - U_{i,j-1}^{k-1/2}}{\Delta z} + \frac{U_{i,j}^{k+1/2} - U_{i,j}^{k-1/2}}{\beta_{i,j}\Delta t} = 0 \\ \frac{V_{i+1/2,j+1/2}^{k-1/2} - V_{i+1/2,j-1/2}^{k-1/2}}{\Delta z} + \frac{V_{i+1/2,j+1/2}^{k+1/2} - V_{i+1/2,j+1/2}^{k-1/2}}{\alpha_{i+1/2,j+1/2}\Delta t} = 0 \end{aligned}, \tag{4}$$

where (i, j) is the space grid index; k is the time grid index.

The three-point backward difference approximation is more accurate (Bancroft, 2008). Using three-point backward difference operators with respect to z and two-point difference with respect to t, system (3) in the Madariaga-Virieux staggered-grid scheme can be written as

$$\begin{aligned} \frac{3U_{i,j}^{k-1/2} - 4U_{i,j-1}^{k-1/2} + U_{i,j-2}^{k-1/2}}{2\Delta z} + \frac{U_{i,j}^{k+1/2} - U_{i,j}^{k-1/2}}{\beta_{i,j}\Delta t} = 0 \\ \frac{3V_{i+1/2,j+1/2}^{k-1/2} - 4V_{i+1/2,j-1/2}^{k-1/2} + V_{i+1/2,j-3/2}^{k-1/2}}{2\Delta z} + \frac{V_{i+1/2,j+1/2}^{k+1/2} - V_{i+1/2,j+1/2}^{k-1/2}}{\alpha_{i+1/2,j+1/2}\Delta t} = 0 \end{aligned}. \tag{5}$$

From the system (4) or (5), particle velocities on the bottom boundary at time $k+1/2$ are calculated from the data at time $k-1/2$. Systems (4) or (5) are for the bottom boundary of the subsurface model, where backward difference approximations are used with respect to z. Similarly, the left boundary and the right boundary of the subsurface model can, respectively, be approximated by forward difference and backward difference approximations with respect to x.

To illustrate the boundary conditions, a subsurface model, which contains a point diffractor in a homogenous medium in $x-z$ plane, is designed. Figure 1a shows the geometry and the P-wave velocities, although the real subsurface model parameters used by the modelling algorithm are densities and Lame coefficients.

Figure 1b and 1c show, respectively, the modelling result of applying two-point and three-point backward difference approximation. The traces on the green line shown in Figure 1b and 1c are, respectively, shown in Figure 1d and 1e. From the figures we can observe that (1) the reflections from bottom boundary are reduced to a level comparable to the diffractor reflections; and (2) the boundary reflections of the three-point approximation are attenuated more than that of the two-point approximation.

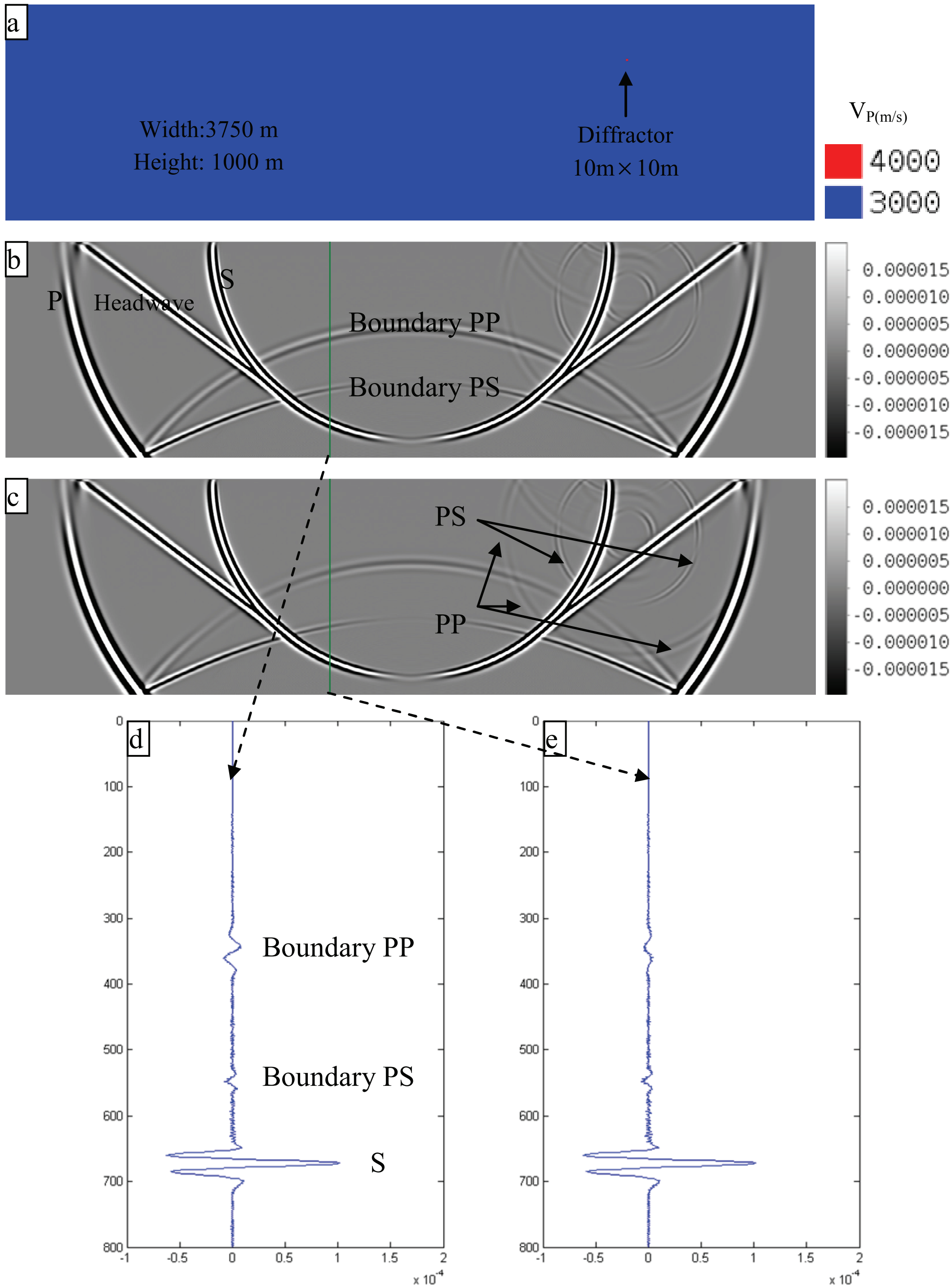

FIG. 1. Point diffractor surbsurface model and the modelling results of absorbing boundary conditions with two-point and three-point backward difference approximations. Figure 1a shows the point diffractor subsurface model; Figure 1b and 1c are, respectively, the results of two-point and three-point approximation on the boundary; Figure 1d and 1e shows, respectively, the traces on the green lines in Figure 1b and 1c.

CONCLUSIONS

The three-point backward difference approximation in absorbing boundary conditions A1, which is the simplest and thus the most popular one of the three absorbing boundary conditions proposed by Clayton and Engquist (1977), results in less reflection from the computational boundaries than the two-point approximation.

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