Generalized frames for Gabor operators in imaging

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ABSTRACT

In numerical wavefield propagation, it is useful to decompose a complex geological region into small local regions of nearly constant velocity, and propagate pieces of the wavefield through each region separately. The total wavefield is then obtained by reassembling all the pieces.

We show here how this decomposition/reassembling is captured mathematically using a windowing procedure which is accurately described by so-called generalized frames. By applying frame theory, we show that a collection of local wavefield propagators combined via a suitable partition of unity, remains a stable propagator, which is a highly desirable property in numerical simulations. These results apply more generally to combinations of linear operators that are useful for many nonstationary filtering operations.

INTRODUCTION

In many geophysical algorithms for seismic imaging, our research teams in CREWES and POTSI have been using Gabor multipliers with non-uniform windows as a basic tool for performing non-stationary filtering, non-stationary deconvolution, and wavefield propagation in inhomogeneous media. The intuition behind this idea is that a complex geological region, such as the salt-dome cross section indicated in Figure 1, can be broken up into a few simpler regions as shown in the middle of the figure, and any physical phenomenon such as wave propagation can be simulated separately in each region, and recombined to produce a final total result.

The basic mathematical tool we use for this involve numerical windows, and algorithms that act separately on the windowed data, followed by perhaps additional windows to refine and recombine the results. While this methodology has led to useful numerical tools, there has been a lack of mathematical theory to describe the details of how the process works, and with it, a lack of theoretical results on the accuracy, stability, and extensibility of the method.

Ultimately what we need are some basic mathematical results that would establish a functional calculus with these operators – that is, show us how to combine operators as products, quotients and exponentials of these window-localized operators, in order to make exact calculations of the physical operators acting in these complex regions. For instance, want to be able to factor the wave equation into one-way waves, and exponentiate to obtain an accurate wavefield propagator. For this we need a sensible functional calculus. We also need control of numerical error, operator norm, and bounds on any approximations that arise.

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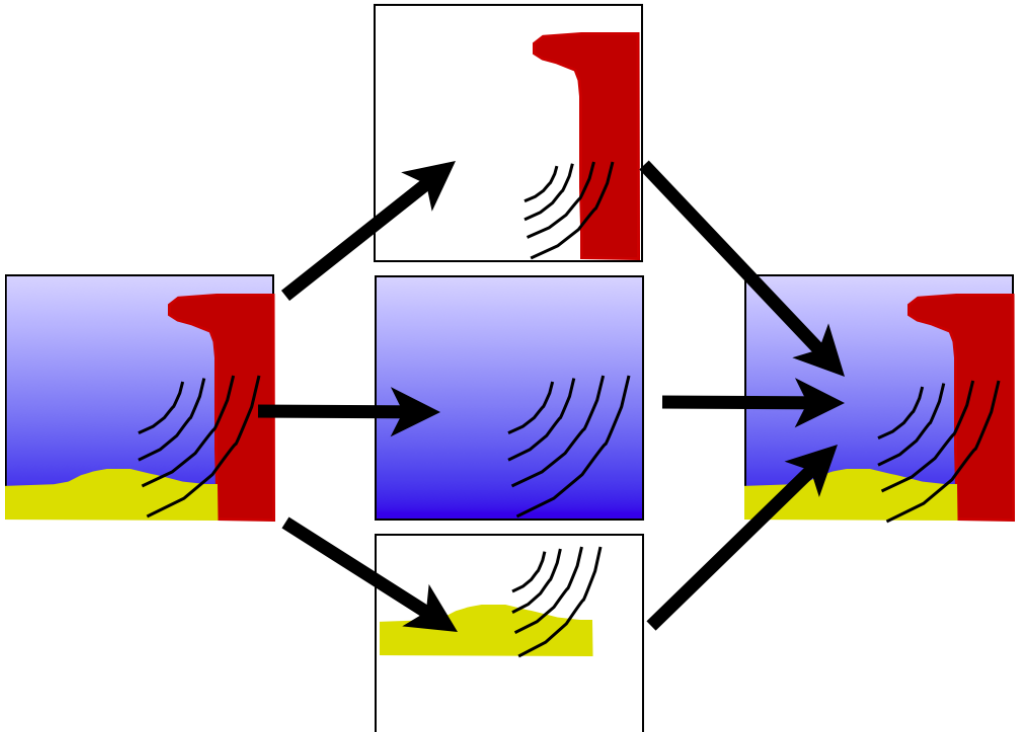

FIG. 1. Propagating a wavefield through a complex medium. Decompose into simpler regions, propagate, reassemble.

Although our methods are based on the ideas of Denis Gabor, the standard Gabor multiplier theory does not apply here, as these depend on uniform windows and group theoretic results concerning modulation and translation operators on a lattice. We don't have uniform windows; we don't have a uniform lattice. So this theory does not apply.

Gabor theory has developed into a branch of frame theory, which is a very rich area of research in numerical methods, especially applied to linear operator theory, such as we use in seismic data processing. It turns out that "generalized frames" is the mathematical theory that can capture the details of our windowing methods. This paper describes how to fit our seismic algorithms into this framework, and in particular we demonstrate that summing up local operators, using a partition of unity, is exactly the process we use and that generalized frames can analyze.

BACKGROUND MODEL FOR DISCUSSION

For this paper, as a seismic application, it is useful to keep one basic model in mind. Frame theory involves Hilbert space and linear operators. For us, a Hilbert space is simply the linear vector space of data. For instance, it could be the set of possible seismic traces we record, or the set of waves that propagate through a seismic section. The linear operators are the things we do to the data with algorithms – for instance, we might want to deconvolve the seismic traces, or we might want to propagate a wavefield in time.

We also window our data – that is, restrict the seismic signal to a particular region in space, say, by multiplying the signal with some window function like a Gaussian. In a collection of seismic traces, for instance, we can mute some traces, and pass others through. Our (non-uniform) windows give rise to multiplication operators $P_1, P_2, P_3, \ldots, P_n$ which act on data f by multiplying it by the various windows. We will see later a partition of unity condition on these winodws, which is just that the sum of squares of the window functions should add up to one, or in operator notation, the squares of the operators P_i sum to the identity:

$$\sum_i P_i^2 = I. \tag{1}$$

When we localize an operator A, we first act on data f with some window operator P_i, then apply A, then P_i again, and sum for the final result. The localized version of A is then a sum

$$A_{loc} = \sum_i P_i A P_i. \tag{2}$$

One goal in this paper is to show that the operator norms behave as we want:

$$||A_{loc}|| \leq ||A||. \tag{3}$$

This is the result that ensures stability of our numerical methods.

More generally, we may want to apply different operators A_i to the different local areas: for instance, in a heterogeneous media, A_1 might represent a wave propagator appropriate for the velocity in region one, A_2 the propagator for the velocity in region two, and so on. In that case, we will show that the norms behave as we want:

$$||\sum_i P_i A_i P_i|| \leq \max_i ||A_i||. \tag{4}$$

In particular, if each A_i is stable (has norm less than one), then so is the combined operator.

In the work below, the products P_i^2 and $P_i A P_i$ will be replaced by products with an adjoint, $P_i^* P_i$ and $P_i^* A P_i$, to make the theory apply to possibly complex-valued windows, and other more general operators. This is standard in mathematical frame theory, so we take advantage of this generality.

GENERALIZED FRAMES (G-FRAMES)

A *frame* is a collection of vectors in a vector space that behaves almost like a basis – every vector in the space can be written as a linear combination of items in the frame, and there are certain non-degeneracy conditions that are specified. We won't point out the details here since we don't intend to use this standard type of frame here, but can point the interested reader to Christensen (2003) for more information. The point of a frame is that it carries some redundant information in the decomposition of the linear space in question, which might be quite useful in the case of lossy communication, or numerical methods that don't know everything.

For our work, generalized frames are what are important. While frames decompose a linear space into the span of one dimensional sets, a generalized space decomposes the linear space into the span of the ranges of several operators. The range can have high dimension – to us, the point is that the range of the operator localizes the data to, say, a homogeneous region of the geology. The precise mathematical definition is as follows:

Definition 1 *A sequence $\{P_1, P_2, \ldots\}$ of bounded linear operators P_i on Hilbert space $\mathcal{H}$ is called a generalized frame (or g-frame) if there are two positive constants A and B such that*

$$A||f||^2 \leq \sum_i ||P_i f||^2 \leq B||f||^2, \qquad \text{for all } f \in \mathcal{H}. \tag{5}$$

Equivalently, the operator sequence is a g-frame if we have the operator inequality

$$AI \leq \sum_i P_i^* P_i \leq BI, \tag{6}$$

where I is the identity operator on $\mathcal{H}$.

The sequence $\{P_1, P_2, \ldots\}$ is called a tight g-frame if $A = B$.

We call $\{P_1, P_2, \ldots\}$ a partition of unity g-frame (POU) if $A = B = 1$.

This definition was introduced in Sun (2006), and generalizes the notion of frames, fusion frames, pseudo-frames, oblique frames, outer frames, and bounded quasi-projectors. Again, we won't discuss all these other types of frames, but refer the interested reader to the bibliography. The inequality constraint in the definition will give a specific algorithm for decomposing the linear space of data into particular components, and then reassembling in a stable manner.

G-frame properties

We copy down here the results we need from the work in Sun (2006). These results are generalizations of the usual frame theory.

Just as with frames, given a generalized frame, one can define the analysis operator, the synthesis operator, and the frame operator.

The analysis operator V maps data vectors f in $\mathcal{H}$ to sequences of vectors in $\mathcal{H} \times \mathcal{H} \times \mathcal{H} \times \cdots$ as

$$Vf = (P_1 f, P_2 f, P_3 f, \ldots). \tag{7}$$

The synthesis operator is simply the adjoint map, mapping sequences in $\mathcal{H}\times\mathcal{H}\times\mathcal{H}\times\cdots$ to $\mathcal{H}$ as

$$V^*(f_1, f_2, f_3, \ldots) = \sum_i P_i^* f_i. \tag{8}$$

The frame operator is the product of the two,

$$S = V^* V = \sum P_i^* P_i. \tag{9}$$

By the g-frame definition, the frame operator is a bounded, invertible positive operator, with

$$AI \leq S \leq BI. \tag{10}$$

By setting $Q_i = P_i S^{-1/2}$, it is easy to check that the sequence $\{Q_1, Q_2, Q_3, \ldots\}$ is a tight g-frame with frame bound 1. In other words, these operators form a partition of unity. In fact this is precisely the construction used in existing numerical methods to create windows that form exact partitions of unity. In particular, this gives a resolution of the identity in the form

$$I = \sum_i Q_i^* Q_i, \tag{11}$$

or in terms of vectors,

$$f = \sum_i Q_i^* Q_i f. \tag{12}$$

Traditionally, researchers often work with the canonical dual frame, obtained by observing the following resolution of the identity:

$$f = SS^{-1}f = S^{-1}Sf = \sum_i P_i^* P_i S^{-1} f = \sum_i S^{-1} P_i^* P_i f. \tag{13}$$

Let $\tilde{P}_i = P_i S^{-1}$, we have two resolutions of the identity from the above, as

$$f = \sum_i P_i^* \tilde{P}_i f = \sum_i \tilde{P}_i^{\,*} P_i f. \tag{14}$$

The main reason to work with the canonical dual is because it gives the minimum norm representation of any vector f in $\mathcal{H}$. Namely, we have this theorem

Theorem 1 *Given a g-frame, a vector f in $\mathcal{H}$, and a sequence of vectors g_i that "synthesize" f in this g-frame (that is, suppose $f = \sum_i P_i^* g_j$), then*

$$\sum_i ||g_i||^2 = \sum_i ||\tilde{P}_i f||^2 + \sum_i ||g_i - \tilde{P}_i f||^2. \tag{15}$$

In other words, the minimum norm synthesis occurs when we choose $g_i = \tilde{P}_i f$.

It is worth pointing out that in many applications, the canonical dual is NOT the one to work this, as it tends to create windows that are jagged and not very smooth, which is a disadvantage for numerical algorithms. In particular, in seismic imaging it is usually better to fulfill the POU condition, rather than trying to use the canonical dual. On the other hand, for other applications such as communication theory, this minimum norm synthesis can be advantageous.

LOCALIZED OPERATOR NORMS

Here we prove a new result:

Theorem 2 *Suppose $\{P_1, P_2, P_3, \ldots\}$ is a g-frame for Hilbert space $\mathcal{H}$, with upper frame constant B, and $\{A_1, A_2, A_3, \ldots\}$ is a sequence of bounded operators on $\mathcal{H}$. Then the localized operator*

$$A_{loc} = \sum_i P_i^* A_i P_i \tag{16}$$

satisfies the operator norm inequatity

$$||A_{loc}|| \leq B \cdot \sup_i ||A_i||. \tag{17}$$

The proof is straightforward, using the analysis and synthesis operators. Let $A_\infty = diag(A_i)$ be the operator on $\mathcal{H} \times \mathcal{H} \times \mathcal{H} \times \cdots$ obtained by acting diagonally, as

$$A_\infty(f_1, f_2, f_3, \ldots) = (A_1 f_1, A_2 f_2, A_3 f_3, \ldots). \tag{18}$$

Clearly, the norm is $||A_\infty|| = \sup_i ||A_i||$. The localized operator is obtained by putting A_∞ between the analysis and synthesis operators, so

$$A_{loc} = V^* A_\infty V \tag{19}$$

and thus we compute norms as

$$||A_{loc}|| \leq ||V^*|| \cdot ||A_\infty|| \cdot ||V|| = ||V^*V|| \cdot ||A_\infty|| = ||S|| \cdot ||A_\infty|| \leq B \cdot \sup_i ||A_i|| \tag{20}$$

and we are done.

In the case where the g-frame forms a partition of unity, then $B = 1$ so we have

$$||A_{loc}|| \leq \sup_i ||A_i||. \tag{21}$$

In the case where the analysis and synthesis windows are different, we get a similar result.

Theorem 3 *Suppose $\{P_1, P_2, P_3, \ldots\}$ and $\{Q_1, Q_2, Q_3, \ldots\}$ are g-frames for Hilbert space $\mathcal{H}$, with upper frame constant B_1, B_2 respectively, and $\{A_1, A_2, A_3, \ldots\}$ is a sequence of bounded operators on $\mathcal{H}$. Then the localized operator*

$$A_{loc} = \sum_i Q_i^* A_i P_i \tag{22}$$

satisfies the operator norm inequality

$$||A_{loc}|| \leq \sqrt{B_1 B_2} \cdot \sup_i ||A_i||. \tag{23}$$

The proof is analogous.

Mathematically, these maps are completely positive maps, so Stinespring's theorem applies. In this special case we can see directly how to represent the complete positive map using isometries in the POU case, which of course is just the analysis/synthesis operators.

The application to seismic data processing is that the local operators may be, say, wavefield propagators. Since each propagator is stable, each A_i has norm equal to one. The above theorem says that the combined operator (the global wavefield operator) will also have norm less than or equal to one, so it too is stable.

STABILITY FOR NONSYMMETRIC WINDOWS

Our numerical methods often use partitions of unity where the analysis and synthesis windows are different. This amounts to having two sets of multipliers P_i, Q_i that satisfy the condition that

$$\sum_i Q_i^* P_i = I. \tag{24}$$

The question is, does the the corresponding localized operator

$$A_{loc} = \sum_i Q_i^* A_i P_i \tag{25}$$

satisfy some reasonable operator norm bounds.

In fact, it may not. It is easy to construct an example where each A_i has norm one, but the summed version A_{loc} has norm about the size of the square root of the number of windows. A summary of the construction goes like this: select the P_i to be windows of width about one, uniformly spaced, that form a partition of unity on their own

$$\sum_i P_i = I. \tag{26}$$

For instance, we could use uniformly spaced Gaussians, suitably normalized. Take the Q_i to be the identity function, corresponding to windows that are each constant one. By the POU condition on the P_i, we get

$$\sum_i Q_i^* P_i = \sum_i P_i = I, \tag{27}$$

so the window pairs P_i, Q_i satisfy the partition of unity condition.

Now, choose the A_i to be a one-way wavefield propagator that moves a wavefield along exactly i steps, in the negative direction. That is, it will move a signal centred in window i along until it is centred in window 0. The sum operator

$$A_{loc} = \sum_{i=1}^{N} Q_i^* A_i P_i \tag{28}$$

will take a sum of isolated signals, centred in windows $1, 2, \ldots, N$ and propagate them all to one big signal centred at window 0. It's not hard to compute that the norm of this operator is $\sqrt{N}$.

So the partition of unity condition, by itself, does not guarantee stability.

A more pertinent question, though, is when do iterates of A_{loc} stabilize? That is, if we build a numerical wavefield propagator, what we really want to know is whether iterating it many times will lead to numerical instabilities. That might be a more interesting question, with possibly a positive result. The point is, the first iterate looks like

$$(\sum_i Q_i^* A_i P_i)(\sum_j Q_j^* A_j P_j), \tag{29}$$

so maybe the mixing of factors $P_i Q_j^*$ in the middle somehow controls the growth.

Unfortunately, no.

The basic answer is that there is no reason that this operator

$$A_{loc} = \sum_i Q_i^* A_i P_i \tag{30}$$

should be stable under iterations, even if the P_i, Q_i form a partition of unity.

We give a one-dimensional counterexample in the following. This might seem unrealistic in the context of multipliers and windows that are higher dimensional, especially in the case of wavefield propagators. However, it really is at the core of any higher dimensional counterexample.

The basic observation is that if you start with two vectors of lengths strictly greater than one, whose inner product is exactly one, then the outer product (a rank one operator) will have norm bigger than one.

So, take $\mathcal{H} = \mathbb{C}$, the one-dimensional complex Hilbert space. Set $P_1 = 1, P_2 = -1, Q_1 = 2, Q_2 = 1$. This give the partition of unity, since

$$Q_1^* P_1 + Q_2^* P_2 = 2 \cdot 1 - 1 \cdot 1 = 1. \tag{31}$$

Our analysis operator and synthesis operators are just

$$V_p = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ and } V_q^* = \begin{bmatrix} 2 & 1 \end{bmatrix}. \tag{32}$$

We have of course $V_q^* V_p = 1$, the frame operator, and when we reverse order, we get

$$V_p V_q^* = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \begin{bmatrix} 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix}, \tag{33}$$

which is a nonselfadjoint idempotent with operator norm $\sqrt{10}$.

The point is that if we try to localize an operator (a_1, a_2), we get

$$A_{loc} = Q_1^* a_1 P_1 + Q_2^* a_2 P_2 = 2a_1 - a_2. \tag{34}$$

So, taking $(a_1, a_2) = (1, -1)$ for the counterexample, we obtain

$$A_{loc} = 3 = 3 \max(|a_1|, |a_2|). \tag{35}$$

So we have picked up a factor of 3, even though we started with a diagonal operator of norm 1. Iterating this operator causes exponential growth of order 3^n, which is very bad.

SOME OPEN QUESTION

The counter-example above used only self-adjoint, commuting, one-dimensional operators, which is a very good counterexample.

However, the elements of the partition of unity were not positive – we have some negative signs in there. In practice, our partitions of unity usually only use positive multipliers. Does this matter? Perhaps it does, so our immediate plan is to try to figure out in the special case of non-negative windows, do we get stability for iterations of wavefield operators, even when we don't have symmetric windows.

Another thing to notice is that the counterexample constructed using wavefield propagators relied very much on the operators "moving around" the signals, from one window to another. Perhaps things are better when we have operator that don't move around so much. For instance, differential operators, and in general pseudo-differential operators don't move the support of a signal – so we should ask the question, for this class of operators, is the POU condition enough. Again, as pseudodifferential operators are important to seismic signal processing (eg. in deconvolution), it is useful to resolve this issue as well.

In particular, we are developing a functional calculus for Gabor multipliers analogous to the theory of pseudodifferential operators, for use in the solution of physical problems modelled by partial differential equations, as discussed in Lamoureux et al. (2008). We expect the theory of generalized frames is exactly the mathematical tool needed to move forward with this development.

SUMMARY

Generalize frames form the mathematical theory for describing the windowing process used in numerical algorithms that decompose a complex geological medium into regions of local homogeneity, where local algorithms can be applied and recombined. We use this theory to show the partition of unity condition on numerical windows allow control of operator norms and control on stability for numerical algorithms such as wavefield propagators. Future work will be to use the the generalized frame theory to make estimates on errors for these algorithms, based on a functional calculus for Gabor multipliers that tell us how to model the solution to physical PDEs using these localized operators.

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