Estimation of Q and phase velocity using the stress-strain relaxation spectrum

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ABSTRACT

The article presents a numerical inversion method for estimation of Q-factor and phase velocity in linear, viscoelastic, isotropic media using reconstruction of relaxation spectrum from measured or computed complex velocity or complex modulus of the medium. Mathematically the problem is formulated as an inverse spectral problem for reconstruction of spectral measure in the analytic Stieltjes representation of the complex modulus using rational approximation. A rational (Padé) approximation to the spectral measure is derived from a constrained least squares minimization problem with regularization. The recovered stress-strain relaxation spectrum is applied to numerical calculation of frequency dependent Q-factor and frequency dependent phase velocity for known analytical models of a standard linear viscoelastic solid (Zener) model as well as a nearly constant-Q model which has a continuous spectrum. Numerical results for these analytic models show good agreement between theoretical and predicted values and demonstrate the validity of the algorithm. The proposed method can be used for evaluating relaxation mechanisms in seismic wavefield simulation of viscoelastic media. The constructed lower order Padé approximation can be used for determination of the internal memory variables in TDFD numerical simulation of viscoelastic wave propagation.

INTRODUCTION

We present a method to recover relaxation spectrum of the medium given measurements of complex velocity or complex viscoelastic modulus, and to further estimate the quality Q-factor and phase velocity. We formulate the problem as an approximation to the spectral measure in the Stieltjes representation of the complex modulus using rational (Padé) approximation. The method of construction of Padé approximation is based on constrained least squares minimization algorithm, regularized by the constraints derived from the analytic Stieltjes representation of the complex modulus. Solution of the constrained minimization problem gives us coefficients of a rational approximation to the spectral measure of the medium. This rational approximation is transformed into Padé approximation by partial fraction decomposition. The method can use as data the values of measured, or simulated (or desired) complex modulus or complex velocity in certain interval of frequencies. The recovered lower order rational ($[p, q]$-Padé) approximation can be used for determination of the internal memory variables in TDFD numerical simulation of viscoelastic wave propagation. The developed technique together with finite difference modeling may eventually lead to an alternative formulation for numerical simulation of viscoelastic wave propagation. The present approach may suggest a new simultaneous inversion technique for estimation of the frequency dependent complex velocities, Q-factors and phase velocities in anelastic attenuating media from vertical seismic profile (VSP) data in geophysics prospecting.

ANALYTIC REPRESENTATION OF VISCOELASTIC MODULUS

We consider a plane compressional wave propagating in a homogeneous isotropic viscoelastic medium with constant material properties. The equation of motion and the relation between stress σ and strain ε for one-dimensional (1D) linear viscoelastic media are represented by

$$\varrho\frac{\partial^2 u}{\partial t^2}=\frac{\partial\sigma}{\partial x},\quad \sigma=\mathcal{M}*\mathrm{d}\varepsilon=\int_{-\infty}^{t}\mathcal{M}(t-\tau)\mathrm{d}\varepsilon(\tau),\quad \varepsilon=\frac{\partial u}{\partial x} \tag{1}$$

where ϱ is the mass density, $u(x,t)$ is the displacement, $\mathcal{M}(t)$ is the relaxation function or medium modulus. In the frequency domain the relation between stress σ and strain ε can be formulated as

$$\sigma(\omega)=M(\omega)\varepsilon(\omega) \tag{2}$$

where $M(\omega)$ is the complex viscoelastic modulus and ω is the angular frequency. The complex velocity and the phase velocity are given by (Carcione, 2007)

$$V(\omega)=\sqrt{\frac{M(\omega)}{\varrho}},\qquad \frac{1}{c(\omega)}=\mathrm{Re}\left[\left(\frac{\varrho}{M(\omega)}\right)^{1/2}\right] \tag{3}$$

respectively. The quality factor Q as a function of ω is defined as

$$Q(\omega)=\frac{\mathrm{Re}M(\omega)}{\mathrm{Im}M(\omega)}=\cot\theta(\omega) \tag{4}$$

where $\theta(\omega)$ is the phase of M. $M(\omega)$ is uniquely determined by a given $Q(\omega)$ in a causal medium since ReM and ImM must obey Kramers-Kronig dispersion relations (Carcione, 2007). The Q-factor characterizes the phase delay between the oscillating stress and strain. In seismic applications, Q is normally assumed to be frequency-independent or only slowly varying with frequency (Kjartansson, 1979). Q-factor is commonly used for evaluating the absorbtion and attenuation of the seismic wave.

Recall the integral expression for the viscoelastic modulus M (Day and Minster, 1984)

$$M(\omega)=M_U-\delta M\int_0^\infty\frac{\mathrm{d}\eta(x)}{\mathrm{i}\omega+x},\quad \text{where }\mathrm{d}\eta(x)=\Phi(-\mathrm{ln}x)\mathrm{d}x,\quad \mathrm{i}=\sqrt{-1}. \tag{5}$$

The non-negative distribution $\Phi(\mathrm{ln}\tau)$ is called the normalized relaxation spectrum of the medium with $\tau=x^{-1}$ being the relaxation time. Here M_U is the unrelaxed modulus and δM is the relaxation of the modulus, respectively. In terms of $M(\omega)$, it is seen from (5) that M_U, the relaxed modulus M_R, and δM are given by

$$M_U=\lim_{\omega\to\infty}M(\omega),\;\; M_R=\lim_{\omega\to 0}M(\omega),\;\; \delta M=M_U-M_R. \tag{6}$$

It is convenient to introduce a new complex variable $s=\mathrm{i}\omega$ and define a new function $G(s)=(M_U-M(s/\mathrm{i}))/\delta M$ which is the integral part of the complex modulus $M(\omega)$ defined in (5). The function G can be written as

$$G(s)=\frac{M_U-M(s/\mathrm{i})}{\delta M}=\int_0^\infty\frac{\mathrm{d}\eta(x)}{s+x},\quad s\in\mathbb{C}\setminus(-\infty,0]\ \text{ with }\ \int_0^\infty\frac{\mathrm{d}\eta(x)}{x}=1 \tag{7}$$

where $\mathrm{d}\eta(x)$ is the non-negative Stieltjes measure on $(0, \infty)$ which characterizes the relaxation spectrum of the medium. The function $G(s)$ is analytic outside the negative real semiaxis in the complex s-plane, all its singular points are in the interval $(-\infty, 0)$. The real-valued function $\eta(x)$ is uniquely determined if it is chosen such that $\eta(x) = \eta(x^+),\ \ \eta(0) = 0$, and it can be obtained from the function G by the Stieltjes inversion formula (Widder, 1946)

$$\eta(x) = \frac{\eta(x^+) + \eta(x^-)}{2} = -\frac{1}{\pi} \lim_{y \to 0^+} \int_0^x \mathrm{Im}\, G(-\nu + \mathrm{i}y)\mathrm{d}\nu. \tag{8}$$

Because the complex velocity $V(\omega)$ in (3) or complex viscoelastic modulus $M(\omega)$ in (5) is frequency dependent, the measurements of $V(\omega)$ or $M(\omega)$ at certain frequencies should be able to provide the desired data set. The function $\eta(x)$ can be approximated by a step function with a finite number of steps (Zhang and Cherkaev, 2008, 2009; Zhang et al., 2009), so that

$$\mathrm{d}\eta(x) \simeq \mathrm{d}\hat{\eta}(x) = \sum_{n=1}^{q} A_n \delta(x + \rho_n)\mathrm{d}x, \ \ x \in (0, \infty) \tag{9}$$

where $A_n > 0$ and $-\infty < \rho_q < ... < \rho_1 < 0$. The function $\eta(x)$ can be approximated by

$$\eta(x) = \int_0^{x^+} \mathrm{d}\eta(t) \simeq \hat{\eta}(x) = \int_0^{x^+} \mathrm{d}\hat{\eta}(t) = \sum_{n=1}^{q} A_n H(x + \rho_n), \ \ x \in (0, \infty) \tag{10}$$

where $H(x)$ is the Heaviside step function. The function $\eta(x)$ defined for $x \in (0, \infty)$ is a non-decreasing, non-negative function corresponding to the Stieltjes function $G(s)$. Thus, the approximation $\hat{G}(s)$ of the function $G(s)$ is given by

$$G(s) \simeq \hat{G}(s) = \sum_{n=1}^{q} \frac{A_n}{s - \rho_n}, \ \ \text{s.t.} \ \ -\infty < \rho_n < 0, \ \ 0 < \frac{A_n}{|\rho_n|} < 1, \ \ \sum \frac{A_n}{|\rho_n|} = 1. \tag{11}$$

Here ρ_n is the n-th simple pole on the negative real semiaxis with positive residue A_n, q is the total number of poles. It follows from (7), (9) and (11) that the approximation of the complex modulus $M(\omega)$ is given by

$$M(\omega) \simeq M_U - \delta M \sum_{n=1}^{q} \frac{A_n}{\mathrm{i}\omega - \rho_n}. \tag{12}$$

Equation (12) gives an expression of discrete approximation of the complex modulus $M(\omega)$ in a partial fraction form. The real parameters A_n and ρ_n in this representation contain all information about the relaxation spectrum of the medium. It follows from (4) and (12), that the Q-factor, the complex velocity $V(\omega)$, and the phase velocity $c(\omega)$ can be estimated in terms of A_n and ρ_n as

$$Q(\omega) \simeq \mathrm{Re}[M_U - \delta M \sum_{n=1}^{q} A_n/(\mathrm{i}\omega - \rho_n)]/\mathrm{Im}[M_U - \delta M \sum_{n=1}^{q} A_n/(\mathrm{i}\omega - \rho_n)], \tag{13}$$

$$V(\omega) \simeq V^c(\omega) = \frac{1}{\sqrt{\varrho}} \left\{ M_U - \delta M \sum_{n=1}^{q} \frac{A_n}{\mathrm{i}\omega - \rho_n} \right\}^{\frac{1}{2}}, \quad c(\omega) \simeq \frac{1}{\mathrm{Re} V^c(\omega)}, \tag{14}$$

respectively.

The partial fraction approximation (12) for the complex modulus $M(\omega)$ implies the relationship between the stress σ and strain ε in the time domain as shown in (Day and Minster, 1984) as

$$\sigma(t) = M_U \left[\varepsilon(t) - \sum_{n=1}^{q} \zeta_n(t) \right] \tag{15}$$

where ζ_n $(n = 1, 2, ..., q)$ are the internal memory variables which satisfy the first-order differential equations (Day and Minster, 1984; Emmerich and Korn, 1987)

$$\frac{\mathrm{d}\zeta_n(t)}{\mathrm{d}t} - \rho_n \zeta_n(t) = A_n \frac{\delta M}{M_U} \varepsilon(t), \quad (n = 1, 2, ..., q). \tag{16}$$

Equation (15) represents the stress σ as a sum of the elastic part $M_U \varepsilon(t)$ and an anelastic part given by the internal memory variable functions $\zeta_n(t)$ $(n = 1, 2, ..., q)$. Substituting equation (15) into equations (1) results in the system of governing differential equations

$$\varrho \frac{\partial^2 u}{\partial t^2} = M_U \left[\frac{\partial^2 u}{\partial x^2} - \sum_{n=1}^{q} \vartheta_n(x, t) \right] \tag{17}$$

where $\vartheta_n(x, t) = \frac{\partial \zeta_n}{\partial x}(x, t)$ satisfies

$$\frac{\mathrm{d}\vartheta_n}{\mathrm{d}t} - \rho_n \vartheta_n = A_n \frac{\delta M}{M_U} \frac{\partial^2 u}{\partial x^2}, \quad (n = 1, 2, ..., q). \tag{18}$$

Comparing to 1D heterogeneous viscoelastic equations (1) the convolution integrals are eliminated in the system of equations (17) by introducing a sequence of variables ϑ_n, with each satisfying a first-order differential equation in time. Equations (18) have to be solved for the unknown functions $\vartheta_n(x, t)$ in addition to the elastic wave equations (17) of motion in the time-domain numerical simulation of wave propagations using finite-difference methods (Krebes and Quiroaa-Goode, 1994; Blanch et al., 1995). The accuracy of numerical computation of wave propagation in an attenuating medium depends on how well the poles ρ_n and residues A_n of the function $G(s)$ are determined when using equations (18). From a practical computation point of view, it is important (crucial) to keep the number of internal memory variable functions ζ_n in (16) or ϑ_n in (18) as low as possible. This yields to construct a lower order rational approximation of complex modulus $M(\omega)$ for modeling of Q-factor in the frequency domain.

Let us assume that the complex velocity $V(\omega)$ or complex modulus $M(\omega)$ can be measured or computed in a range of frequencies or can be modeled for a specific viscoelastic material. We describe an inversion method below which allows us to identify the real parameters A_n and ρ_n, and to construct a rational ($[p, q]$-Padé) approximation of $M(\omega)$ in (12), especially for a lower order $[p, q]$-Padé approximation of $M(\omega)$ from measured or computed complex velocity. Therefore, the Q-factor can be evaluated using formula (13).

RATIONAL APPROXIMATION FOR INVERSION

We note that the function $G(s)$ has a discrete approximation $\hat{G}(s)$ of the partial fraction form (11). Therefore, the right hand side of the first equation in (11) can be approximated by a rational ($[p, q]$-Padé) function given by (Baker Jr. and Graves-Morris, 1996):

$$\hat{G}(s) = \sum_n \frac{A_n}{s - \rho_n} = \frac{a_0 + a_1 s + a_2 s^2 + \cdots + a_p s^p}{b_0 + b_1 s + b_2 s^2 + \cdots + b_q s^q} \qquad (p < q) \tag{19}$$

where a_l $(l = 0, 1, \ldots, p)$ and b_k $(k = 0, 1, \ldots q)$ are real coefficients to be determined, respectively. Let us suppose that the function $G(s)$ has at least one pole, and all the poles of the denominator in (19) are simple. Since the poles ρ_n of the function $G(s)$ lie in the interval $(-\infty, 0)$, we normalize the polynomial coefficient $b_0 = 1$ in the denominator of (19) which allows us to identify the nonzero poles of G. To derive the linear system of equations for the coefficients a_l's and b_k's in (19), we further assume that the measured data pairs (s_j, g_j) $(j = 1, 2, .., N)$ of the function G are given: $g_j = G(s_j)$, $s_j = \mathrm{i}\omega_j$, and $\mathrm{i} = \sqrt{-1}$. Here ω_j is the frequency sample data point and N is the total number of the complex measured values of $G(s)$. We require that the constructed approximation $\hat{G}(s)$ agreed with the measured values of $G(s)$ at the points s_j. Then equation (19) can be written as

$$\frac{a_0 + a_1 s_j + a_2 s_j^2 + \cdots + a_p s_j^p}{1 + b_1 s_j + b_2 s_j^2 + \cdots + b_q s_j^q} = g_j \tag{20}$$

where a_l $(l = 0, \ldots, p)$, b_k $(k = 1, \ldots, q)$ are required unknown coefficients. Equation (20) is equivalent to the following system

$$a_0 + a_1 s_j + \cdots + a_p s_j^p - b_1 g_j s_j - b_2 g_j s_j^2 - \cdots - b_q g_j s_j^q = g_j, \ \ (j = 1, 2, \ldots, N). \tag{21}$$

Therefore, the system (21) for the unknown coefficients a_l's and b_k's can be further expressed as the following system:

$$\mathbf{Sc} := \begin{pmatrix} 1 & s_1 & \cdot\cdot & s_1^p & -g_1 s_1 & -g_1 s_1^2 & \cdot\cdot & -g_1 s_1^q \\ 1 & s_2 & \cdot\cdot & s_2^p & -g_2 s_2 & -g_2 s_2^2 & \cdot\cdot & -g_2 s_2^q \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ 1 & s_N & \cdot\cdot & s_N^p & -g_N s_N & -g_N s_N^2 & \cdot\cdot & -g_N s_N^q \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \cdot\cdot \\ a_p \\ b_1 \\ \cdot\cdot \\ b_q \end{pmatrix} = \begin{pmatrix} g_1 \\ g_2 \\ \cdot\cdot \\ g_N \end{pmatrix} = \mathbf{g} \tag{22}$$

It is clear that in order for the Padé coefficients a_l's and b_k's to be uniquely determined, the total number of the measurements is required to be greater or equal to the number of coefficients, i.e., $N \geq p + q + 1$. The reconstruction problem of determining the column real coefficient vector $\mathbf{c} = [a_0, a_1, \cdots, a_p, b_1, b_2, \cdots, b_q]^\top$ in (22) is an inverse problem. It is ill-posed and requires regularization to develop a stable numerical algorithm.

To construct a real solution vector $\mathbf{c}$ of $[p, q]$-Padé coefficients for the inverse problem (22), we use complex matrices $\mathbf{S} = \mathbf{S}_\mathrm{R} + \mathrm{i}\mathbf{S}_\mathrm{I}$ and $\mathbf{g} = \mathbf{g}_\mathrm{R} + \mathrm{i}\mathbf{g}_\mathrm{I}$ where subindices r and i indicate the real and imaginary parts of the matrices with entries in terms of data.

We introduce a penalization term in the Tikhonov regularization functional $\mathcal{T}^{\lambda}(\mathbf{c}, \mathbf{g}_{\mathrm{R}}, \mathbf{g}_{\mathrm{I}})$ (Tikhonov and Arsenin, 1977), so that the problem (22) can be formulated as the following constrained least squares minimization problem with the regularization parameter $\lambda > 0$ chosen properly (Zhang and Cherkaev, 2008, 2009; Zhang et al., 2009):

$$\min_{\mathbf{c}} \mathcal{T}^{\lambda}(\mathbf{c}, \mathbf{g}_{\mathrm{R}}, \mathbf{g}_{\mathrm{I}}) = \min_{\mathbf{c}} \{ \mid\mid \mathbf{S}_{\mathrm{R}}\mathbf{c} - \mathbf{g}_{\mathrm{R}} \mid\mid^2 + \mid\mid \mathbf{S}_{\mathrm{I}}\mathbf{c} - \mathbf{g}_{\mathrm{I}} \mid\mid^2 + \lambda^2 \mid\mid \mathbf{c} \mid\mid^2 \}$$
$$\text{subject to} \quad -\infty < \rho_n < 0, \quad 0 < \frac{A_n}{|\rho_n|} < 1, \quad n = 1, 2, \dots, q. \tag{23}$$

Here $||\cdot||$ is the usual Euclidean norm, parameters A_n and ρ_n in the constraints (23) are residues and poles of the partial fractions decomposition of the reconstructed $[p, q]$-Padé approximation of $G(s)$. To find the minimizer of the problem (23), we solve its Euler equation; the solution is given by

$$\mathbf{c} = \{\mathbf{S}_{\mathrm{R}}^{\top}\mathbf{S}_{\mathrm{R}} + \mathbf{S}_{\mathrm{I}}^{\top}\mathbf{S}_{\mathrm{I}} + \lambda \mathbf{I}_{p+q+1}\}^{-1}\{\mathbf{S}_{\mathrm{R}}^{\top}\mathbf{g}_{\mathrm{R}} + \mathbf{S}_{\mathrm{I}}^{\top}\mathbf{g}_{\mathrm{I}}\} \tag{24}$$

where $\mathbf{I}_{p+q+1}$ denotes the $(p+q+1)\times(p+q+1)$ identity matrix. After reconstruction of the real coefficient vector $\mathbf{c}$ of the rational function approximation $\hat{G}(s)$, its decomposition into partial fractions (19), gives $[p, q]$-Padé approximation of $G(s)$. The reconstructed function $\hat{G}(s)$ can be used to identify the relaxation spectrum for a viscoelastic medium and to estimate the quality Q-factor for such a medium using formula (13).

SPECTRAL REPRESENTATION OF STANDARD LINEAR SOLID MODEL

We consider the time-dependent relaxation function of stress-strain relation in a standard linear solid (SLS) (Generalized Zener) model (Carcione et al., 1988)

$$\mathcal{M}(t) = M_R \left[1 - \sum_{n=1}^{L} \left(1 - \frac{\tau_{\epsilon_n}}{\tau_{\sigma_n}} \right) \mathrm{e}^{-t/\tau_{\sigma_n}} \right] H(t), \quad \tau_{\epsilon_n} \geq \tau_{\sigma_n} \tag{25}$$

where τ_{ϵ_n}, τ_{σ_n} denote material strain relaxation time and stress relaxation time for the n-th mechanism, respectively. This model was also introduced in (Blanch et al., 1995; Tal-Ezer et al., 1990; Liu et al., 1976) in order to obtain a nearly constant quality Q-factor over the seismic frequency range of interest. Here the relaxed modulus $M_R = M_U - \delta M$, L is the number of relaxation mechanisms, and $H(t)$ is the heaviside step function. The unrelaxed modulus is obtained for $t = 0$ in (25) as in the following:

$$M_U = M_R \left[1 - \sum_{n=1}^{L} \left(1 - \frac{\tau_{\epsilon_n}}{\tau_{\sigma_n}} \right) \right]. \tag{26}$$

Applying the Laplace transform in s-multiplied form (Day and Minster, 1984):

$$F(s) = s \int_0^{\infty} \mathcal{F}(t) \mathrm{e}^{-st} \mathrm{d}t \tag{27}$$

to the stress-strain relation (25), and setting $s = \mathrm{i}\omega$, the complex modulus can be derived as

$$M(\omega) = M_R \left[1 - \sum_{n=1}^{L} \left(1 - \frac{\tau_{\epsilon_n}}{\tau_{\sigma_n}} \right) \frac{\mathrm{i}\omega}{\mathrm{i}\omega + \tau_{\sigma_n}^{-1}} \right]. \tag{28}$$

Noticing (26) and the definition of function $G(s)$ in (7), corresponding to the complex modulus $M(\omega)$ (28), $G(s)$ is found in the following L-term partial fractions form

$$G(s) = \left(\frac{M_U}{M_R} - 1\right)^{-1} \sum_{n=1}^{L} \frac{(\tau_{\epsilon_n}/\tau_{\sigma_n} - 1)\tau_{\sigma_n}^{-1}}{s + \tau_{\sigma_n}^{-1}}. \tag{29}$$

In the complex s-plane, equation (29) implies a representation for the poles and residues of the function $G(s)$:

$$\rho_n = -\tau_{\sigma_n}^{-1}, \quad A_n = (M_U/M_R - 1)^{-1}[(\tau_{\epsilon_n}/\tau_{\sigma_n} - 1)\tau_{\sigma_n}^{-1}], \qquad (1 \le n \le L). \tag{30}$$

The location of poles and residues of the function G depends on the strain-stress relaxation parameters τ_{ϵ_n} and τ_{σ_n}. From equation (26), one can check that the residues A_n and poles ρ_n in (30) satisfy the sum rule property as in the last equation of (11).

Let us assume that the complex velocity $V(\omega)$ or the complex modulus $M(\omega)$ can be simulated in a range of frequencies for the standard linear solid model (28) and the real parameters A_n and ρ_n can be recovered using the reconstruction algorithm of a $[p, q]$-Padé approximation of the function $G(s)$ as described in the previous sections. From equations (30), we can calculate the strain-stress relaxation parameters τ_{ϵ_n} and τ_{σ_n} in terms of the recovered poles ρ_n and residues A_n of $G(s)$ explicitly as in the following:

$$\tau_{\epsilon_n}^c = (M_U/M_R - 1)A_n\rho_n^{-2} - \rho_n^{-1}, \quad \tau_{\sigma_n}^c = -\rho_n^{-1}, \qquad (1 \le n \le q). \tag{31}$$

where the superscript c indicates the computed value of τ_{ϵ_n} and τ_{σ_n}. From formulas (30) and (31), we can see that the parameters τ_{ϵ_n} and τ_{σ_n} can be simply calculated once the poles ρ_n and residues A_n of the approximation $\mathrm{d}\hat{\eta}(x)$ of the spectral measure $\mathrm{d}\eta(x)$ are determined. For the standard linear solid model (28), by the definition (4) and equation (28), the Q-factor as a function of frequency ω can be estimated for different lower orders $q \le L$ using the derived equivalent formulas

$$Q^c(\omega) = \frac{1 + \displaystyle\sum_{n=1}^{q} \frac{(\tau_{\epsilon_n}^c - \tau_{\sigma_n}^c)\omega^2\tau_{\sigma_n}^c}{1 + \omega^2(\tau_{\sigma_n}^c)^2}}{\displaystyle\sum_{n=1}^{q} \frac{(\tau_{\epsilon_n}^c - \tau_{\sigma_n}^c)\omega}{1 + \omega^2(\tau_{\sigma_n}^c)^2}} \quad \text{or} \quad Q^c(\omega) = \frac{M_R - \delta M \displaystyle\sum_{n=1}^{q} \frac{A_n\rho_n^{-1}\omega^2}{\omega^2 + \rho_n^2}}{\delta M \displaystyle\sum_{n=1}^{q} \frac{A_n\omega}{\omega^2 + \rho_n^2}} \tag{32}$$

where $Q^c(\omega)$ represents the calculated quality factor Q. The complex modulus and complex velocity can be calculated as

$$M^c(\omega) = M_R\left[1 - \sum_{n=1}^{q}\left(1 - \frac{\tau_{\epsilon_n}^c}{\tau_{\sigma_n}^c}\right)\frac{\mathrm{i}\omega}{\mathrm{i}\omega + (\tau_{\sigma_n}^c)^{-1}}\right] \tag{33}$$

and

$$V(\omega) \simeq V^c(\omega) = \frac{\sqrt{M^c(\omega)}}{\varrho}, \tag{34}$$

respectively. Therefore, the phase velocity can be calculated using (14).

NUMERICAL EXAMPLES

Results for the standard linear solid model

In the following numerical simulations we employ the values of material strain relaxation time τ_{ϵ_n} and stress relaxation time τ_{σ_n} shown in Table 1 (refer to (Tal-Ezer et al., 1990) where these values were used for numerically solving the 1-D viscoelastic equation of motion with relaxation mechanisms) to calculate the synthetic complex viscoelastic modulus $M(\omega)$ with $L = 5$ relaxation mechanisms to yield a constant quality factor $Q = 100$ at 50 data points in the seismic exploration band of frequencies from 2Hz to 50Hz.

Table 1. True values of relaxation times for five mechanisms to yield a constant $Q = 100$ ($\omega = 2 \sim 50$Hz) for the synthetic viscoelastic modulus given in (Tal-Ezer et al., 1990).

n	1	2	3	4	5
τ_{ϵ_n}(s)	0.3196389	0.0850242	0.0226019	0.0060121	0.0016009
τ_{σ_n}(s)	0.3169863	0.0842641	0.0224143	0.0059584	0.0015823

The poles and normalized residues of the spectral function $G(s)$ were reconstructed using the developed inversion algorithm for different orders of $q = p + 1$: (1) $q = 7$, (2) $q = 4$ and (3) $q = 3$ when there is no noise in the data. The location of the recovered poles of the function G is located on the negative real semiaxis, the analytically and numerically calculated normalized spectral measure $\xi(x)$ and $\hat{\xi(x)}$ are shown in Fig. 1. The recovered five poles and residues of the function $\mathrm{d}\eta$ corresponding to the synthetic modulus $M(\omega)$ when $q = 7$ are reconstructed very accurately with the computed sum $\sum A_n/|\rho_n| \approx 1.0000000$. For the other two cases poles and residues are reconstructed with the calculated sum $\sum A_n/|\rho_n| \approx 0.9736328$ for $q = 4$ and $\sum A_n/|\rho_n| \approx 0.8855121$ for $q = 3$, respectively. The recovered poles and residues of G are further used to convert the values of material strain relaxation time τ_{ϵ_n} and stress relaxation time τ_{σ_n} using formulas (31) with the number of relaxation mechanisms being less than 5. These reconstructed values of τ_{ϵ_n} and τ_{σ_n} are shown in Table 2 when there is no noise in the data using the lower order $q < 5$ in $[p, q]$-Padé approximant method. In Table 2, $\tau^{\mathrm{c}}_{\epsilon_n}$ and $\tau^{\mathrm{c}}_{\sigma_n}$ stand for the predicted relaxation times for $L = q = 3$ and $L = q = 4$ mechanisms to yield a constant $Q = 100$ ($\omega \in 2\pi[2, 50]$Hz) using the constrained Padé approximant method.

Table 2. Recovery of relaxation times for three and four mechanisms to yield a constant $Q = 100$ ($\omega = 2 \sim 50$Hz) using the constrained Padé approximant method ($\tau^{\mathrm{c}}_{\epsilon_n}$, $\tau^{\mathrm{c}}_{\sigma_n}$ stand for the predicted values of relaxation times).

n	1	2	3	4	q
$\tau^{\mathrm{c}}_{\epsilon_n}$(s)	0.1508046	0.0247268	0.0060695	0.0016029	4
$\tau^{\mathrm{c}}_{\sigma_n}$(s)	0.1486096	0.0244912	0.0060145	0.0015842	
$\tau^{\mathrm{c}}_{\epsilon_n}$(s)	0.0597205	0.0075747	0.0016859		3
$\tau^{\mathrm{c}}_{\sigma_n}$(s)	0.0586769	0.0074939	0.0016649		

To estimate the frequency dependent quality Q-factor and the frequency dependent phase velocity for the SLS model, we chose the density $\varrho = 2000\mathrm{kg/m}^3$ and the relaxed modulus $M_R = 8$Gpa in the numerical simulations. Fig. 2 shows the results of the recovered Q-factors (left) and phase velocity $c(\omega)$ (right). It is seen from the left part of Fig. 2 that the estimated values of the Q-factors are nearly constant $Q = 100$ over the frequency band between 12Hz and 37Hz for the lower order $[p, q]$-Padé approximant method. However, the calculated Q-factors were not very good approximation to the constant $Q = 100$ in both low frequency range about $2 \sim 12$Hz and high frequency range between 37Hz and 50Hz for this particular used analytic SLS model. The estimated Q-factors shown in the left Fig. 2 are calculated using equivalent formulas (32). The true and computed phase velocity versus frequency shown on the right of Fig. 2 were calculated using formula (14). The results of computations for Q-factors and phase velocity agree with the true values in published simulations of (Tal-Ezer et al., 1990), especially for values at high frequencies. The calculated values of relaxation mechanisms can be used for seismic wavefield simulation in viscoelastic media.

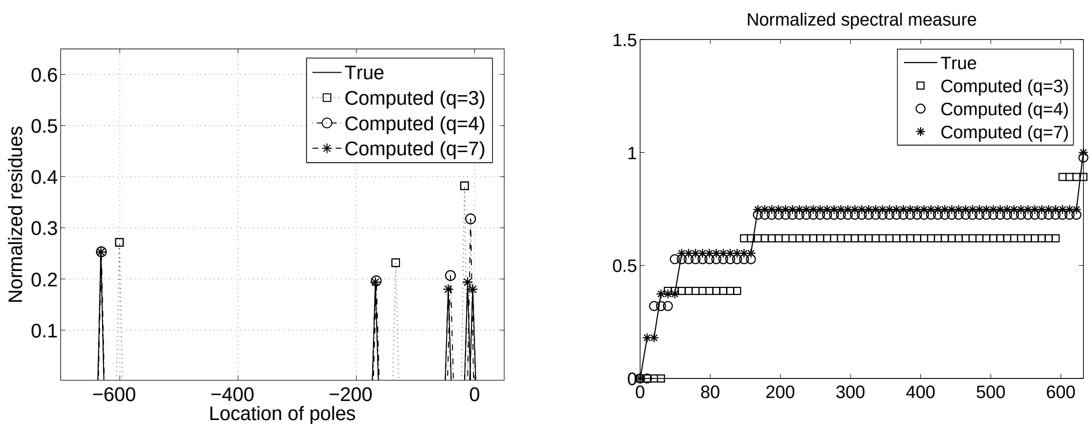


FIG. 1. Reconstruction of residues and poles of the function $G(s)$ (left) and the spectral measure $\eta(x)$ (right).

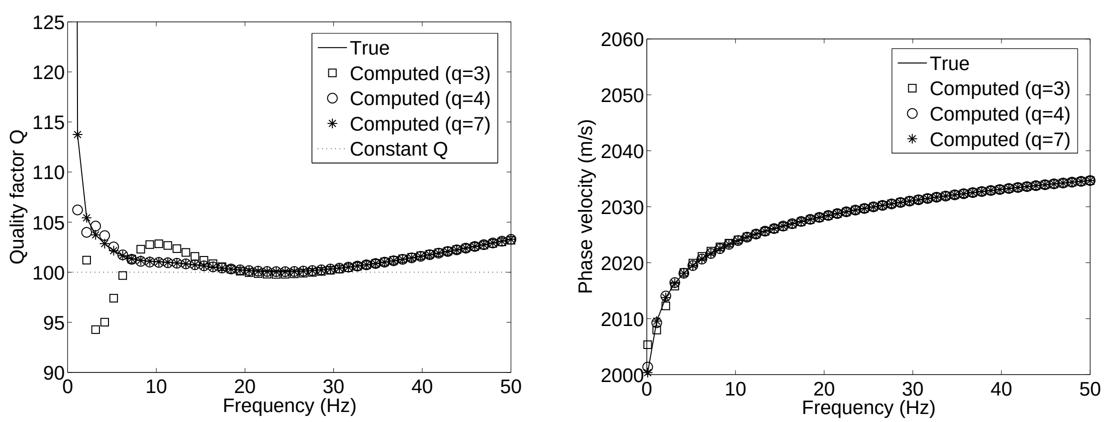


FIG. 2. True and computed quality Q-factor (left) and phase velocity $c(\omega)$ (right).

Results for a nearly constant-Q model with a continuous spectrum

To further examine the effectiveness of the developed inversion method we consider a nearly constant-Q model with a continuous relaxation spectrum. The synthetic spectral measure for this model has a constant relaxation spectrum in the finite interval $[x_0, x_1] \subset (0, \infty)$ $(x_0 \neq x_1)$ given by

$$\frac{\mathrm{d}\eta(x)}{\mathrm{d}x} = \begin{cases} \left[\ln\left(\frac{x_1}{x_0}\right)\right]^{-1}, & \text{if } x_0 \leq x \leq x_1 \\ 0, & \text{if } x > x_0 \text{ or } x < x_1. \end{cases} \tag{35}$$

The representation of the normalized spectral measure is derived as

$$\xi(x) = \int_0^{x^+} \frac{\mathrm{d}\eta(t)}{t} = \begin{cases} 0, & \text{if } 0 \leq x < x_0 \\ \left[\ln\left(\frac{x_1}{x_0}\right)\right]^{-1} \ln\left(\frac{x}{x_0}\right), & \text{if } x_0 \leq x \leq x_1 \\ 1, & \text{if } x > x_1. \end{cases} \tag{36}$$

It is easy to check the normalized spectral measure $\mathrm{d}\xi(x) = \mathrm{d}\eta(x)/x$ in (36) satisfies the sum rule property in (7). The corresponding function G defined in (7) can be derived analytically as

$$G(s) = \int_0^{\infty} \frac{\mathrm{d}\eta(x)}{s+x} = \left[\ln\left(\frac{x_1}{x_0}\right)\right]^{-1} \ln\left(\frac{s+x_1}{s+x_0}\right), \qquad s = \mathrm{i}\omega. \tag{37}$$

The corresponding complex modulus and complex velocity are obtained as

$$M(\omega) = M_U - \delta M \left[\ln\left(\frac{x_1}{x_0}\right)\right]^{-1} \ln\left(\frac{\mathrm{i}\omega + x_1}{\mathrm{i}\omega + x_0}\right), \tag{38}$$

and

$$V(\omega) = \left[\frac{1}{\varrho}\left(M_U - \delta M \left[\ln\left(\frac{x_1}{x_0}\right)\right]^{-1} \ln\left(\frac{\mathrm{i}\omega + x_1}{\mathrm{i}\omega + x_0}\right)\right)\right]^{\frac{1}{2}} \tag{39}$$

respectively, where ϱ is the density. The unrelaxed modulus $M_U = \varrho c_U^2$, c_U is the unrelaxed velocity, and the relaxation modulus $\delta M = M_U - M_R$, $M_R = \varrho c_R^2$, c_R is the relaxed velocity.

In the numerical experiments we chose $\varrho = 2400\mathrm{kg/m}^3$, $c_U = 3500\mathrm{m/s}$, $c_R = 3000\mathrm{m/s}$, which gives $\delta M = 7.8\mathrm{Gpa}$. The complex velocity measurements were simulated at 50 data points in a range frequency as $\omega \in 2\pi[10^{-2}, 10^2]\mathrm{s}^{-1}$, and the interval $[x_0, x_1] = [0.35, 10^4]$ was chosen for the support of the normalized spectral measure $\mathrm{d}\xi(x)$. The recovered normalized residues and poles of the function $G(s)$ for the continuous relaxation spectrum model are shown in the left part of Fig. 3 using the Padé approximants of different orders (1) $p - 1 = q = 4$ and (2) $p - 1 = q = 5$ when there is no noise in the data. We compared analytically and numerically calculated normalized spectral measure functions. The right part of Fig. 3 shows the true normalized spectral measure $\xi(x)$ and the approximation $\hat{\xi}(x)$ of the normalized spectral measure. The location of the recovered four poles with the computed sum $\sum A_n/|\rho_n| \approx 0.9413156$ for $q = 4$ and five poles with $\sum A_n/|\rho_n| \approx 0.9569588$ for $q = 5$ lays in between -10000 and -0.35 on the negative real semiaxis.

The recovered complex modulus and complex velocity of the continuous spectrum model were further used to estimate the Q-factor and the phase velocity $c(\omega)$ over the frequency band between 10^{-2}Hz and 10^2Hz using formulas (13) and (14). Fig. 4 represents the reconstruction of the quality factor Q (left) and the phase velocity (right) of the model versus frequency using the Padé approximants of two different orders. It is seen from the left part of Fig. 4 that the quality factor of the continuous spectrum model is almost nearly constant $Q \simeq 21$ and the recovered quality factors for $q = 4$ and $q = 5$ fits the frequency dependent Q-values of the model very well in a range of frequency about $0.2\text{Hz} \sim 100\text{Hz}$. The phase velocity increases with frequency illustrated in the right part of Fig. 4. The true and reconstructed phase velocities also fits fairly well in the frequency band from 10^{-2}Hz to 10^{2}Hz.

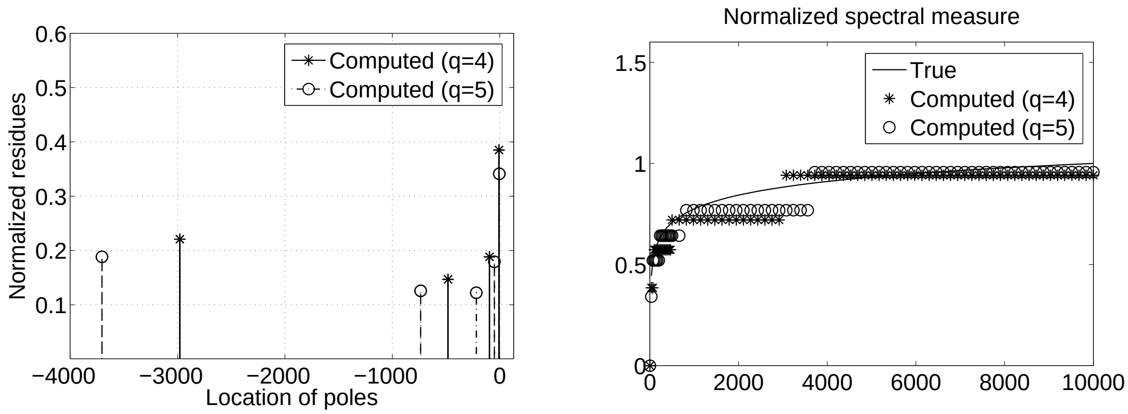


FIG. 3. Reconstruction of poles and residues of the normalized spectral measure $\xi(x)$ (left) and the normalized spectral measure $\xi(x)$ (right).

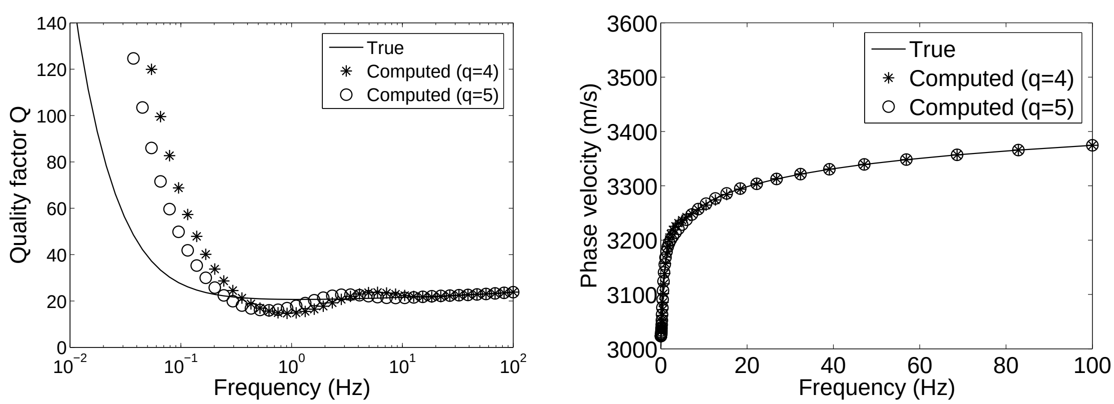


FIG. 4. Calculation of quality Q-factors (left) and phase velocities $c(\omega)$ (right) for model with a continuous spectral measure using different orders of Padé approximation.

We also calculated values of relative errors for the estimation of phase velocity, quality Q-factor, complex velocity, and complex modulus at sample data points over the given frequency band demonstrated in Table 3. In Table 3, the frequency band for calculation of relative errors of phase velocity, complex velocity, and complex modulus is from 0.01Hz to 100Hz, and $0.2\text{Hz} \sim 100\text{Hz}$ for the Q-factors. The relative error formula follows

$$E_0 = \max_{1 \le j \le N} \left[\frac{|g(\omega_j) - \hat{g}(\omega_j)|}{|g(\omega_j)|} \right] \tag{40}$$

where ω_j $(j = 1, 2, ..., N)$ are the frequency sample data points and N is the total number of sample data. Here $g(\omega)$ represents the given frequency dependent function and $\hat{g}(\omega)$ the approximate function of $g(\omega)$, respectively.

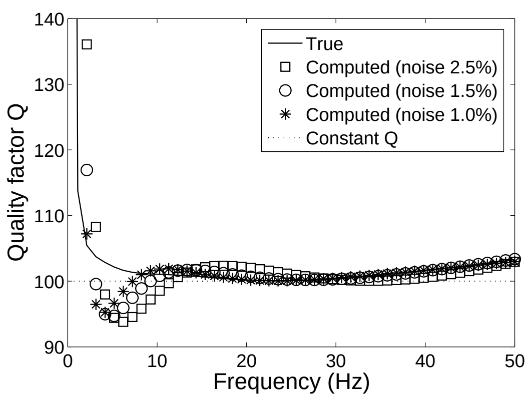
Table 3. Calculated relative errors of true and computed physical parameters using formula (40).

Error	Phase velocity	Quality factor	Complex velocity	Complex modulus	q
E_0	1.2344e-02	4.7875e-01	1.2349e-02	2.4545e-02	4
E_0	9.1699e-03	2.3328e-02	9.1676e-03	1.8251e-02	5

Sensitivity analysis of the method

To numerically illustrate the sensitivity analysis of the estimated quality Q-factors and phase velocities, a uniformly distributed random noise was calculated as percentage of exact value at each measured data point of the synthetic SLS model. We have performed numerical experiments to examine the sensitivity of the algorithm for different noise levels added to the input data. The order $q = p - 1 = 5$ in the inversion algorithm was chosen to reconstruct the spectral function $G(s)$ and the strain-stress relaxation time parameters using data with added noise.

In the numerical experiments any recovered pole ρ_n of $G(s)$ that is off the negative real semiaxis is discarded based on the inversion algorithm so that the total number of reconstructed relaxation mechanisms is less than $q = 5$ in each case of data with added noise. The reconstructed stress-strain relaxation time parameters were used to evaluate the Q-factors and phase velocities. Fig. 5 illustrates the estimation of Q-factors (left) and phase velocities (right) for data with $1.0\%, 1.5\%$ and 2.5% noise. The results of numerical computations show that even with added noise, the computed Q-factors are nearly constant $Q = 100$ in the frequency band about $\omega = 10 \sim 40$Hz, and the recovered phase velocities agree with the true values in the frequency range about $\omega = 2 \sim 50$Hz, which demonstrates the stability of the reconstruction.



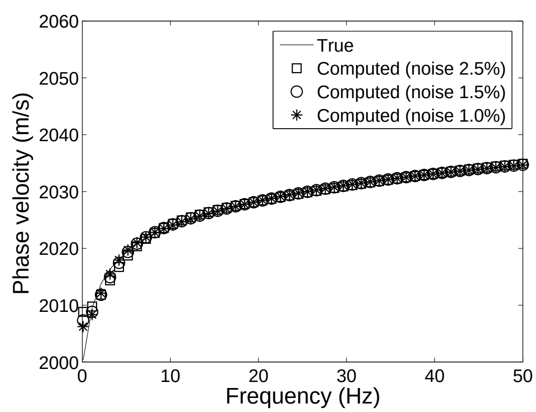


FIG. 5. True and computed quality factor Q (left) and phase velocity $c(\omega)$ (right) for data with $1.0\%, 1.5\%$ and 2.5% noise.

CONCLUSIONS

We developed a new numerical inversion method for estimation of Q-factor and phase velocity in homogeneous dissipating media using Padé approximation. The approach is based on rational ($[p, q]$-Padé) approximation of the spectral measure in the Stieltjes representation of the complex viscoelastic modulus, which contains all information about relaxation spectrum of the medium. The problem is formulated as a constrained least squares minimization problem with regularization constraints provided by the Stieltjes representation of the complex modulus. The method was tested using analytical models of viscoelastic media with a continuous spectrum as well as a standard linear solid (Zener) model. The numerical results demonstrate the effectiveness of the developed approach. The method can be used for identification of relaxation parameters of viscoelastic materials from measurements of complex velocity or complex modulus. The recovered relaxation mechanisms can be used for seismic wavefields simulation in viscoelastic media. Our approach may provide significant savings in the computer memory and computation time needed for numerical simulation of seismic wave propagations in viscoelastic media.

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