Gabor multipliers for pseudodifferential operators and wavefield propagators

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ABSTRACT

We summarize some applications of Gabor multipliers as a numerical implementation of certain linear operators that arise in seismic data processing, including differential operators, nonstationary filters, and wavefield propagators. We demonstrate an approximation formula for pseudodifferential operators using Gabor multipliers. We present a demonstration of the almost factorization of Gabor symbols that is used in a fundamental way for nonstationary deconvolution. We give a numerical example of wavefield propagation through the EAGE salt model.

INTRODUCTION

Gabor multipliers are a general class of linear operators acting on numerical representations of physical signals, that are useful for a variety of signal processing and mathematical modelling applications. As a nonstationary extension of Fourier multipliers, the Gabor multiplier is a useful tool for approximating differential operators with non-constant coefficient functions, time- or spatial-varying filters, and evolution operators for physical processes in inhomogeneous media.

Just as Fourier multipliers are built on the Fourier transform, the Gabor multiplier is built on the Gabor transform, which is a time-frequency representation of a given signal or function of time. Given a signal $f(t)$, its Gabor transform $\mathcal{G}f(t,\omega)$ is a function of two variables that specifies the spectral energy present in the signal near time t at frequency ω. This representation can be understood as a short time (or windowed) Fourier transform of the signal, typically given in the form

$$\mathcal{G}f(t,\omega) = \int_{-\infty}^{\infty} f(s)g(s-t)e^{-2\pi i s\omega}\, ds, \tag{1}$$

where the window function $g(s-t)$ is fixed to localize the signal near time t.

The Gabor transform extracts more useful information from a signal that evolves in time, as it can identify changes in spectral contents as the signal progresses. For example, compare the two representations in Figures 1 and 2. The first figure shows a Vibroseis sweep with the usual Fourier transform representation, indicating a frequency content from 2Hz to 100Hz. The second figure shows how that frequency content evolves over time – in the time-frequency display at the bottom of Figure 2, we see at the beginning of the signal, that the energy content is at the low end of the frequency band. As time progresses, the energy content increases linearly in frequency, indicated by the climbing ramp in the figure.

As this example shows, the Gabor transform produces a more detailed representation

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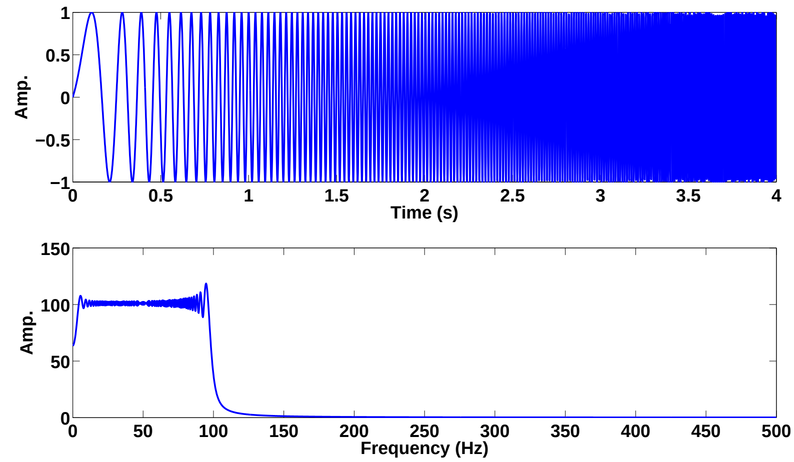

FIG. 1. A synthetic Vibroseis sweep, 2 to 100 Hz, and FFT.

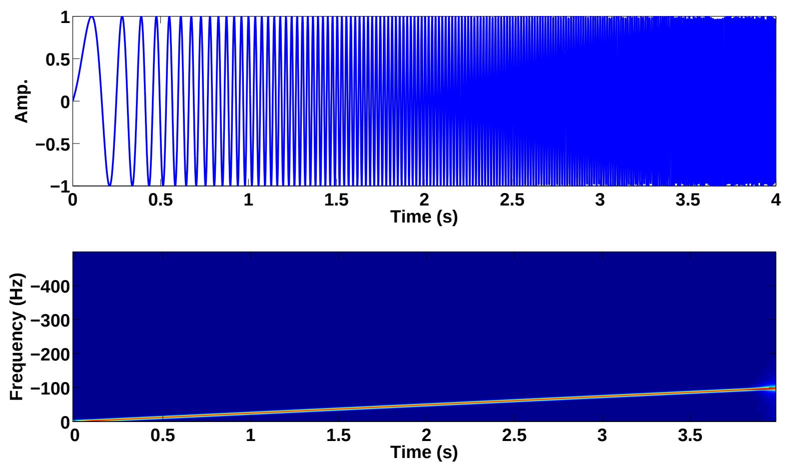

FIG. 2. A synthetic Vibroseis sweep, 2 to 100 Hz, and Gabor transform.

of the signal as it evolves in time. The Gabor multiplier is an operator that acts directly on this more detailed representation of the signal, and as a result, the Gabor multiplier can make a more precise modification of the the signal than is possible in the Fourier domain alone.

In its simplest form, a Gabor multiplier simply modifies a signal in the time-frequency domain by multiplying its Gabor representation $\mathcal{G}f(t,\omega)$ by a fixed function $\alpha(t,\omega)$, and returning to the time domain with the adjoint of the Gabor transform. Denoting this Gabor multiplier as G_α, we can express this operation as the application of three simpler linear operators acting on signal f in the form

$$G_\alpha f = \mathcal{G}^* M_\alpha \mathcal{G} f. \tag{2}$$

Here $\mathcal{G}$ is the Gabor transform operator, $\mathcal{G}^*$ its adjoint, and M_α the operation of multiplication by the function α in the t-ω space.

By a judicious choice of window $g(s-t)$ and sampling appropriately in both the time and frequency domain, we obtain a fast numerical algorithm that can be implemented using the Fast Fourier Transform, and with an approximate functional calculus that allows us to compose Gabor multipliers through combination of different multipliers α. (See Lamoureux et al. (2008); Lamoureux and Margrave (2009).)

As an example of a simple Gabor multiplier, consider modifying the signal shown in Figure 3, which represents a Vibroseis sweep that includes the third harmonic. In order to eliminate this harmonic, we modify the signal in the Gabor domain by setting to zero the terms in the upper ramp that represent the harmonic. That is, we choose the multiplier α to take values zero on the harmonic, and take values one on the fundamental.

The multiplier α is illustrated by its data plot in Figure 4 and the results of applying this multiplier to the sweep with harmonic is shown in Figure 5. As we see in the figure, the odd harmonic has been smoothly eliminated.

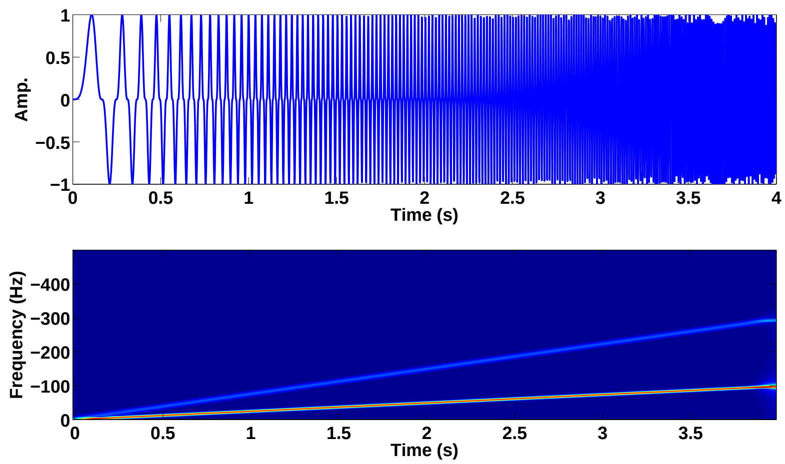

FIG. 3. A synthetic Vibroseis sweep, 2 to 100 Hz, with an odd harmonic.

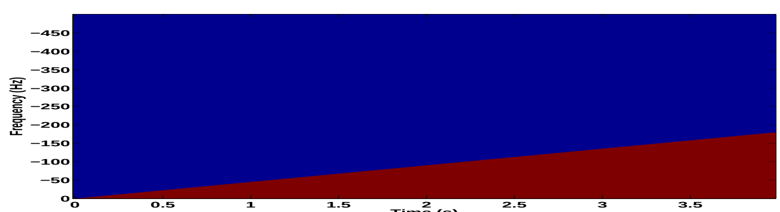

FIG. 4. The permissible region of frequencies to pass, in t-ω domain.

It is worth noting that with a stationary filter (eg. a Fourier multiplier), we could remove the harmonics above 100 Hz, say. However, this would not be adequate to remove the lower frequency odd harmonics in the first two seconds of the signal. Thus the nonstationary behaviour of the Gabor multiplier is better suited to harmonic removal in this time-variant signal.

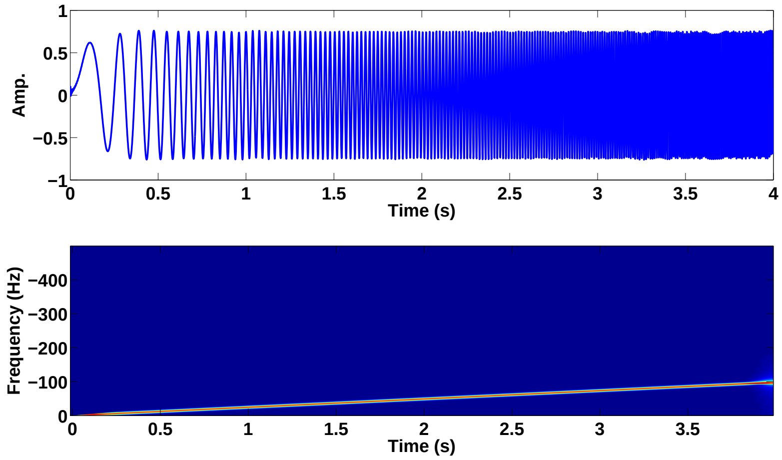

FIG. 5. The result of non-stationary filtering: harmonic is removed.

GENERALIZED GABOR MULTIPLIERS AND G-FRAMES

By replacing the translated window function $g(s-t)$ by a discrete family of windows $g_k(s)$, one obtains a more general Gabor transform defined by the formula

$$\mathcal{G}f(t_k,\omega) = \int_{-\infty}^{\infty} f(s)g_k(s)e^{-2\pi i s\omega}\,ds, \tag{3}$$

where the point t_k is the center of mass of window g_k. Such a transform is useful when it is not necessary to have a uniform partition of the signal space: we use a large window for regions where the signal is relatively stationary, and small regions where the characteristics of the signal are changing quickly.

A good choice of windows will ensure that the adjoint operator $\mathcal{G}^*$ is a left inverse for the generalized Gabor transform. Indeed, the condition that

$$\mathcal{G}^*\mathcal{G}f = f, \text{ for all signals } f\ , \tag{4}$$

is equivalent to the partition of unity condition

$$\sum_k |g_k(t)|^2 = 1, \text{ for all } t \text{ in the support of } f. \tag{5}$$

It is sometime convenient to use one set of windows for the forward transform $\mathcal{G}$ and another set for the inverse (adjoint) transform $\mathcal{H}^*$ with

$$\mathcal{H}f(t_k,\omega) = \int_{-\infty}^{\infty} f(s)h_k(s)e^{-2\pi i s\omega}\,ds. \tag{6}$$

In this case, the requirement that the adjoint operator $\mathcal{H}^*$ is a left inverse for $\mathcal{G}$,

$$\mathcal{H}^*\mathcal{G}f = f, \text{ for all signals } f\ , \tag{7}$$

is equivalent to requiring a similar partition of unity condition, that

$$\sum_k \overline{h_k(t)} g_k(t) = 1, \text{ for all } t \text{ in the support of } f. \tag{8}$$

In this case, the Gabor multiplier determined by symbol $\alpha(t, \omega)$ is given as the operator

$$G_\alpha f = \mathcal{H}^* M_\alpha \mathcal{G} f, \text{ for all signals } f. \tag{9}$$

It is straightforward to verify that this multiplier may be written as a sum of operators (summing over the windows), with

$$G_\alpha = \sum_k M_{hk}^* C_{\alpha k} M_{gk}, \tag{10}$$

where M_{gk} is simply multiplication by the window function $g_k(t)$, M_{hk} is multiplication by the dual window function $h_k(t)$, and $C_{\alpha k}$ is a convolution operator (over time) whose frequency response (as a function of frequency ω) is simply $\alpha(t_k, \omega)$.

Thus, the ordinary and generalized Gabor multipliers are sums of localized convolution operators. This is a special case of operators built from a localizing frame.

In general, given a family of (localizing) windows g_k, a set of dual windows h_k satisfying a partition of unity condition, and a family of linear operators A_k, we define the corresponding global operator as

$$A_{global} = \sum_k M_{hk}^* A_k M_{gk}. \tag{11}$$

These global operators have local properties given by the A_k. The mathematical theory of generalized frames describes in detail the properties of such operators. (See Lamoureux and Margrave (2009); Sun (2006).)

PSEUDODIFFERENTIAL OPERATORS

An order-m differential operator (in one dimension) of the form

$$Kf(t) = \sum_{k=0}^{m} a_k(t) \frac{d^k f}{dt^k}(t) \tag{12}$$

can be represented in the form of an integral operator, with

$$Kf(t) = \int_{-\infty}^{\infty} \alpha(t, \omega) \widehat{f}(\omega) e^{2\pi i \omega t} \, d\omega, \tag{13}$$

where

$$\alpha(t, \omega) = \sum_k a_k(t)(-2\pi i \omega)^k \tag{14}$$

is the symbol of the operator, and $\widehat{f}$ is the Fourier transform of the signal f.

More generally, given any function $\alpha(t,\omega)$ of the two variables t,ω, and certain mild conditions on the smoothness and growth of the function at infinity, one can define a pseudodifferential operator K_α on signals f by the formula

$$K_\alpha f(t) = \int_{-\infty}^{\infty} \alpha(t,\omega)\widehat{f}(\omega)e^{2\pi i\omega t}\,d\omega. \tag{15}$$

This operator K_α is called the Kohn-Nirenberg pseudodifferential operator for symbol α.

The symbol α is suggestive of a Gabor multiplier.

In fact, we can show in the special case where the analysis windows g_k are constant 1, the synthesis windows h_k form a partition of unity,

$$\sum_k h_k(t) = 1, \text{ for all } t, \tag{16}$$

and the symbol α is slowly varying with respect to this partition, then the Gabor multiplier G_α is close to the pseudodifferential operator K_α.

To see this, we note that we may approximate the symbol α using the partition of unity, so

$$\begin{aligned} \alpha(t,\omega) &= \sum_k \overline{h_k(t)}\alpha(t,\omega) && (17)\\ &\approx \sum_k \overline{h_k(t)}\alpha(t_k,\omega), && (18) \end{aligned}$$

since the symbol is varying slowly. Thus we may approximate the pseudodifferential operator as

$$\begin{aligned} K_\alpha f(t) &= \int_{-\infty}^{\infty} \alpha(t,\omega)\widehat{f}(\omega)e^{2\pi i\omega t}\,d\omega && (19)\\ &\approx \int_{-\infty}^{\infty} \sum_k \overline{h_k(t)}\alpha(t_k,\omega)\widehat{f}(\omega)e^{2\pi i\omega t}\,d\omega && (20)\\ &= \sum_k \overline{h_k(t)}\int_{-\infty}^{\infty} \alpha(t_k,\omega)\widehat{f}(\omega)e^{2\pi i\omega t}\,d\omega && (21)\\ &= \sum_k M_{hk}^{*}C_{\alpha k}f(t) = G_\alpha f(t), && (22) \end{aligned}$$

where we have noticed the last integral above is a Fourier multiplier with symbol $\alpha(t_k,\omega)$, corresponding to the convolution operator $C_{\alpha k}$ described in the previous section.

Pseudodifferential operators also come in an adjoint form. For symbol α, the adjoint operator is given in terms of the Fourier transform of its output, as

$$\widehat{A_\alpha f}(\omega) = \int_{-\infty}^{\infty} \alpha(t,\omega)f(t)e^{-2\pi i\omega t}\,dt. \tag{23}$$

For these adjoint operators, we may approximate A_α with Gabor multiplier G_α provided the windows g_k form the partition of unity, the h_k are constant 1, and the symbol α is again slowly varying with respect to the partition of unity.

PRODUCT RESULTS FOR DECONVOLUTION

Gabor methods have been used extensively for seismic trace deconvolution (see Margrave and Lamoureux (2001); Margrave et al. (2002, 2004); Montana and Margrave (2006).). A key fact used in the decon algorithm is that the source, reflectivity, and attenuation process approximately factor in the Gabor domain. That is, for seismic data $d(t)$ recorded from a seismic test with source input $s(t)$, reflectivity $r(t)$, and attenuation process given by symbol $\alpha(t,\omega)$, we expect an approximate factorization in the form

$$\mathcal{G}d(t,\omega) \approx \widehat{s}(\omega)\alpha(t,\omega)\mathcal{G}r(t,\omega), \tag{24}$$

where $\mathcal{G}d, \mathcal{G}r$ are the Gabor transforms of the recorded data and reflectivity respectively, and $\widehat{s}$ is the Fourier transform of the source wavelet.

This approximation has been noticed numerically in our earlier work. We can justify it mathematically using the approximations in the last section. We assume the attenuation symbol $\alpha(t,\omega)$ is slowly varying relative to the partition of unity given by windows g_k, and the source wavelet is concentrated to a small time interval near $t = 0$. These are reasonable physical assumptions in the case of moderate Q-attenuation and an impulse source (eg. dynamite blast).

Our mathematical model for the seismic experiment is that the recorded data $d(t)$ is obtained as the result of an attenuation operators A_α (expressed in the adjoint form of a pseudodifferential operator with symbol α), applied to the relectivity, and then convolved with the source wavelet. That is, we may write the data as the output of a sequence of operations

$$d = s * (A_\alpha r), \tag{25}$$

with $*$ denoting the convolution operator over time.

The Gabor transform for signal d is obtained by taking the Fourier transform of the product of the signal with the k-th window g_k, so we obtain an expression in the time-frequency domain of the Gabor transform

$$\begin{aligned}
\mathcal{G}d(t_k,\omega) &= \widehat{g_k \cdot d}(\omega) && (26)\\
&= (\widehat{g_k} * \widehat{d})(\omega) && (27)\\
&= \widehat{g_k} * (\widehat{s} \cdot \int \alpha(t,\omega)r(t)e^{-2\pi i\omega t}\,dt) && (28)\\
&= \widehat{g_k} * (\widehat{s} \cdot \int \sum_j g_j(t)\alpha(t,\omega)r(t)e^{-2\pi i\omega t}\,dt) && (29)\\
&\approx \widehat{g_k} * (\widehat{s} \cdot \int \sum_j g_j(t)\alpha(t_j,\omega)r(t)e^{-2\pi i\omega t}\,dt) && (30)
\end{aligned}$$

where we have used the Fourier transform to change a product into convolution, and used the assumption that the symbol α is changing slowly with respect to the partition of unity.

Pull out the sum from the integral to obtain

$$\begin{aligned} \mathcal{G}d(t_k,\omega) &\approx \widehat{g_k} * (\widehat{s} \cdot \sum_j \alpha(t_j,\omega) \int g_j(t) r(t) e^{-2\pi i \omega t}\, dt) && (31)\\ &= \sum_j \widehat{g_k} * (\widehat{s} \cdot \alpha(t_j,\omega) \cdot \widehat{g_j \cdot r}) && (32)\\ &= \sum_j \widehat{g_k} * \mathcal{F}(s * a_j * (g_j \cdot r)), && (33) \end{aligned}$$

where we have used the Fourier transform $\mathcal{F}$ to change products to convolutions. We have used the notation $a_j = a_j(t)$ for the inverse Fourier transform of the function $\alpha(t_k,\omega)$, transformed with respect to frequency ω. Next, pull the convolution with $\widehat{g_k}$ into the Fourier transform to obtain

$$\mathcal{G}d(t_k,\omega) \approx \sum_j \mathcal{F}([s * a_j * (g_j \cdot r)] \cdot g_k). \tag{34}$$

Now, using the assumption that the attenuated source wavelet $s * a_j$ has small support, we may pull the factor g_k past the convolution to get a second approximation

$$\begin{aligned} \mathcal{G}d(t_k,\omega) &\approx \sum_j \mathcal{F}([s * a_j * (g_j \cdot r \cdot g_k]) && (35)\\ &= \mathcal{F}([s * a_j * (\sum_j g_j \cdot r \cdot g_k)]) && (36)\\ &= \mathcal{F}([s * a_j * (r \cdot g_k)]), && (37) \end{aligned}$$

where the sum collapses because of the partition of unity condition on the g_j.

Now the Fourier transform converts those convolutions to products, so we obtain

$$\begin{aligned} \mathcal{G}d(t_k,\omega) &\approx \mathcal{F}([s * a_j * (r \cdot g_k)]) && (38)\\ &= \widehat{s}(\omega) \cdot \alpha(t_k,\omega) \cdot \widehat{r \cdot g_k}(\omega) && (39)\\ &= \widehat{s}(\omega) \cdot \alpha(t_k,\omega) \cdot \mathcal{G}r(t_k,\omega) && (40) \end{aligned}$$

where we use again the fact that the Gabor transform of r is just the Fourier transform of the product $r \cdot g_k$.

This verifies the approximate factorization.

Q-ATTENUATION

Gabor deconvolution, discussed in the previous section, models the earth's attenuation process with a pseudodifferential operator whose symbol $\alpha(t,\omega)$ decays exponentially with time and frequency. More precisely, the symbol magnitude is given by

$$|\alpha(t,\omega)| = e^{-t\pi|\omega|/Q}, \tag{41}$$

for a fixed constant Q, positive time t and frequency ω. For discretely sampled signals, we may assume the exponential decay holds only to Nyquist, so we obtain a Fourier series expansion for the log amplitude as

$$\log|\alpha(t,\omega)| = -t\pi|\omega|/Q = -(t\pi/Q)\cdot\sum_{n=0}^{\infty} a_n\cos(\pi n\omega/Nyq), \text{ for } \omega\in[-Nyq,Nyq], \tag{42}$$

where the a_n are the coefficients in the cosine expansion of the even function $|\omega|$.

The phase of the symbol is obtained from the minimum phase assumption, and thus is given as the exponential of the harmonic conjugate of the log amplitude. The conjugate of cosines are sines, so we have explicitly

$$\arg\alpha(t,\omega) = -(t\pi/Q)\cdot\sum_{n=1}^{\infty} a_n\sin(\pi n\omega/Nyq), \text{ for } \omega\in[-Nyq,Nyq]. \tag{43}$$

This phase factor may also be obtained by taking the Hilbert transform of the log amplitude spectrum at fixed time t, which is an alternative method of obtaining the harmonic conjugate.

The symbol is thus determined by combining these two terms, so

$$\alpha(t,\omega) = \exp(-(t\pi/Q)\cdot\sum_{n=0}^{\infty} a_n(\cos(\pi n\omega/Nyq) + i\sin(\pi n\omega/Nyq)). \tag{44}$$

In a numerical implementation, the infinite sum may be replaced by the finite sum of the Fast Fourier Transform. The minimum phase symbol $\alpha(t,\omega)$ can be easily obtained from a cepstral computation, such as in the MATLAB command 'rceps,' in the Signal Processing Toolkit.

In Figure 6 we demonstrate the effect of Q-attenuation on two pulses, one at time delay $t = 1$, another at time $t = 2$. The first pulse is only moderately attenuated, with a slight phase delay. The second pulse has greater attenuation, and a greater phase delay, due to the increased effect of the term $\exp(-t\pi/Q|\omega|)$. These attenuated pulses were computed numerically, using an exact formula phase and amplitude terms in the attenuation operators.

The observed increasing phase delay is a consequence of the mathematical operator implementing the minimum phase Q-attenuation. This may be the source of an observed linear phase error in Gabor deconvolution, discussed in Montana and Margrave (2005).

The Gabor multiplier implementation replaces the symbol $\alpha(t,\omega)$ with a sampled version $\alpha(t_k,\omega)$. The resulting operator is locally minimum phase preserving, but we have no results to show that it is globally so.

It is possible to consider Gabor multipliers that model more general attenuation processes, where the value of Q changes with the window, such as

$$|\alpha(t_k,\omega)| = \exp(-t_k\pi|\omega|/Q_k), \tag{45}$$

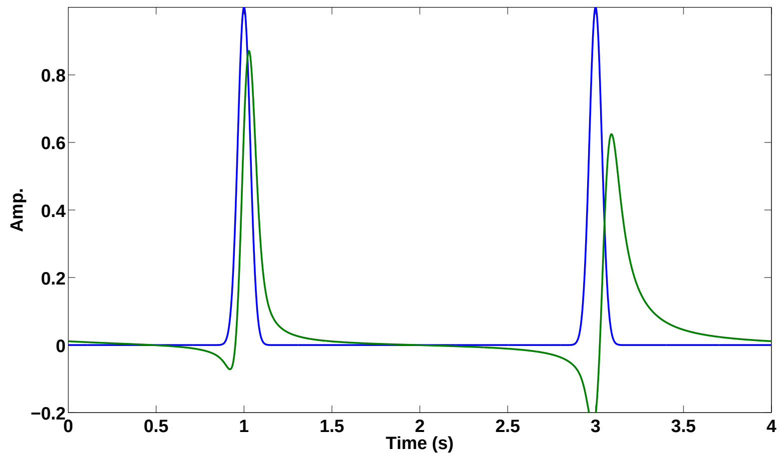

FIG. 6. Two pulses and the result of Q-attenuation.

or where the Q value depends on frequency

$$|\alpha(t_k, \omega)| = \exp(-t_k \pi |\omega| / Q(\omega)). \tag{46}$$

However, we do not have a theory that indicates when such operators are minimum phase preserving, which would be a useful physical constraint on our model.

WAVEFIELD PROPAGATORS

The windowing method of Gabor multipliers may be used to propagate a wavefield through a complicated velocity model, as discussed in Ma and Margrave (2008). As a demonstration of the method, we consider here the case of wave propagation through a salt dome. Figure 7 show the velocity model for a complex geological region around a salt dome, with velocities ranging from a low of 1500 m/s to a high of 4500 m/s. We can approximate this complex region using six spatial windows, each one representing one fixed velocity, for six evenly spaced velocities between 1500 and 4500 m/s. A fixed velocity wavefield propagator is created for each window, and the full wavefield is decomposed by each window, propagated one time step, and the various parts recombined before repeating the window/propagate step again.

Figures 8 through 10 show every tenth step of the iteration. Observe in the last few frames how the presence of the salt dome is indicated by the acceleration and breakup of the wavefront.

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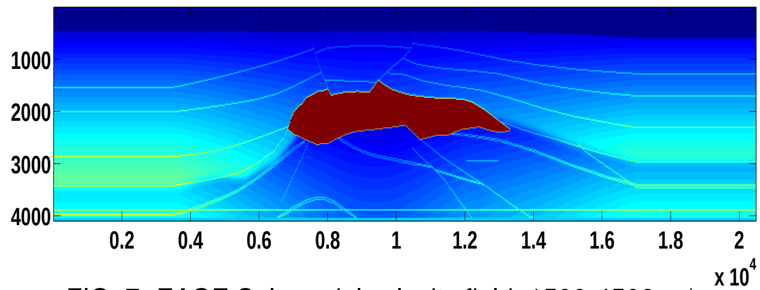

FIG. 7. EAGE Salt model velocity field, 1500-4500 m/s.

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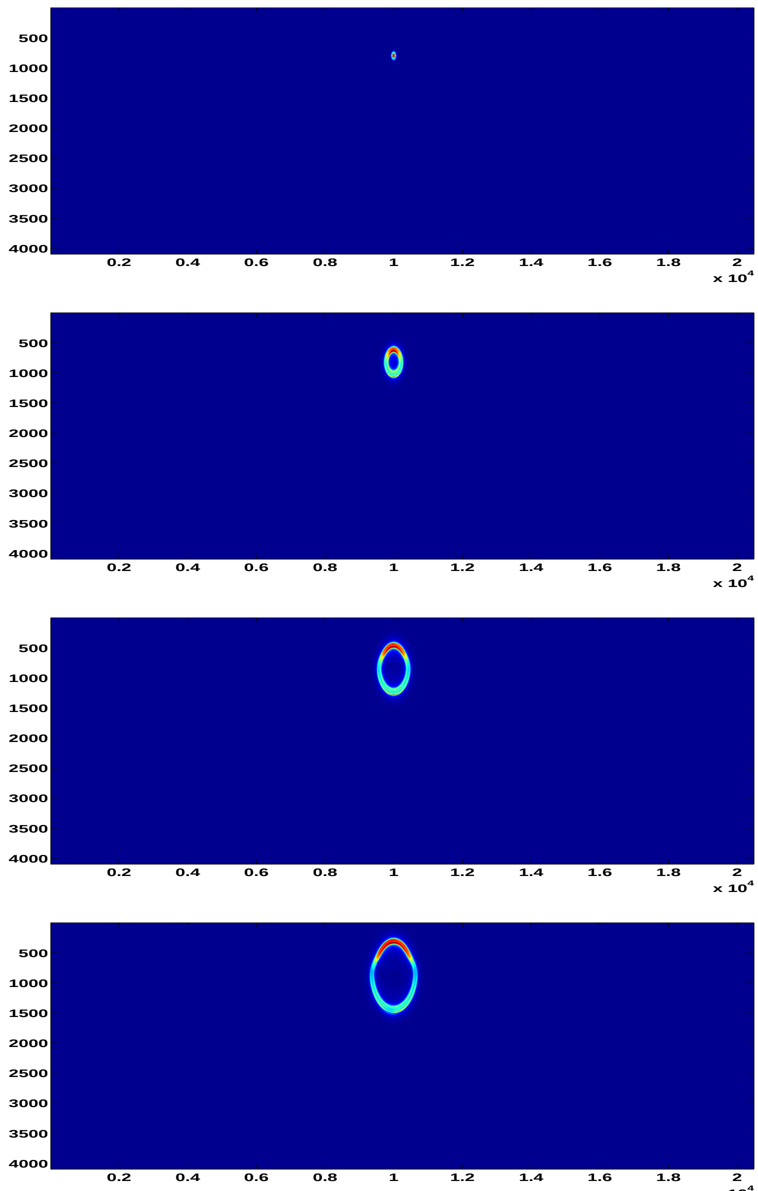

FIG. 8. Numerical simulation of seismic wave propagation through salt model.

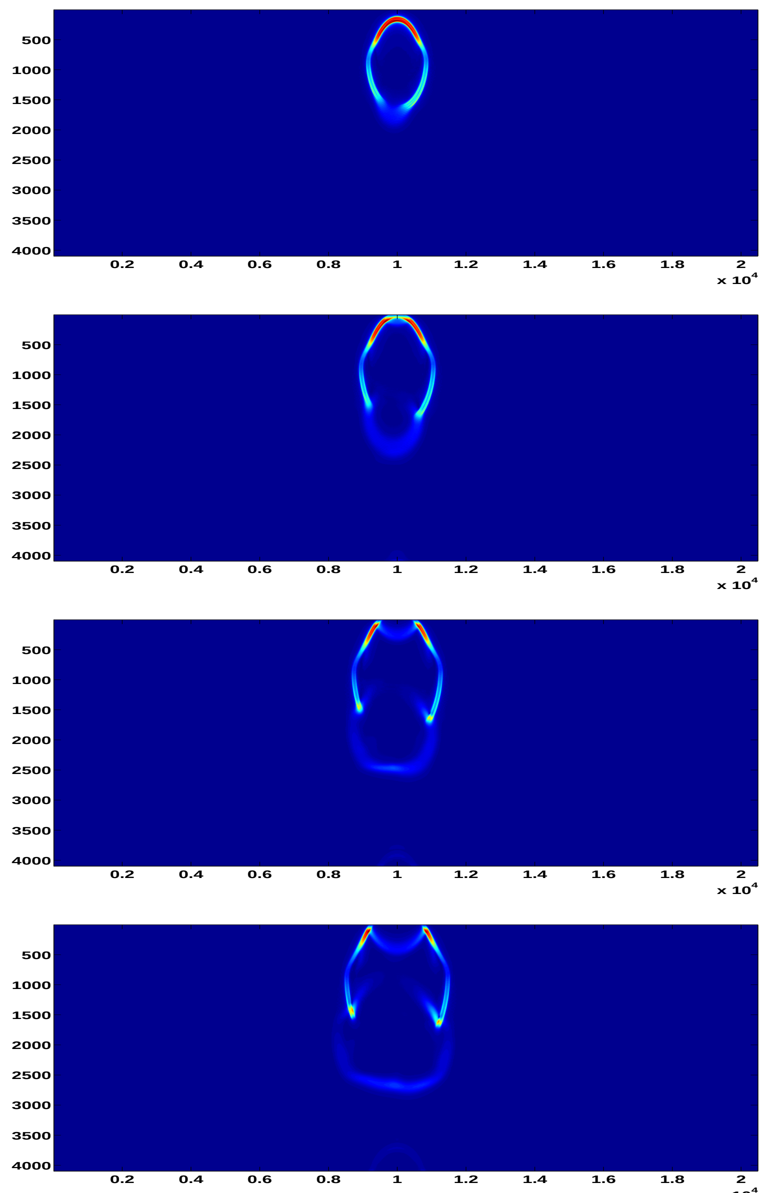

FIG. 9. Numerical simulation of seismic wave propagation through salt model.

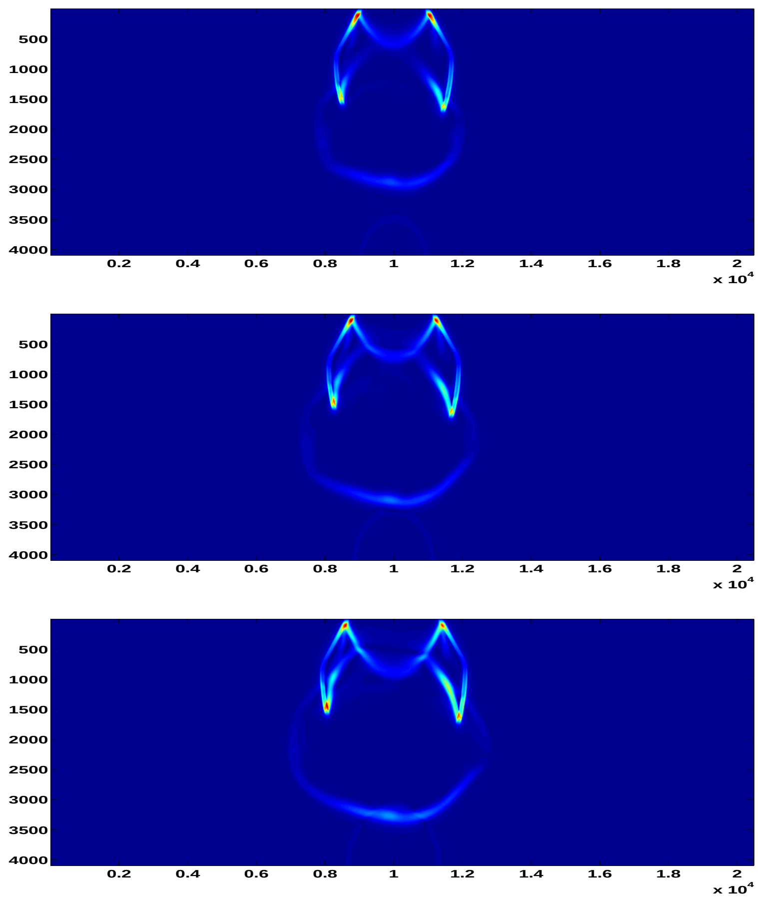

FIG. 10. Numerical simulation of seismic wave propagation through salt model.