Minimum phase and attenuation models in continuous time

Michael P. Lamoureux, Peter C. Gibson*, Gary F. Margrave

ABSTRACT

We extend earlier work on minimum phase attenuation models in discrete-time signal processing to the continuous time setting, where real physical processes occur. This includes the propagation of seismic energy through the earth and allows for the modelling of attenuation processes.

Minimum phase signals are characterized by an energy condition, equivalent to an outer function identification in the complex half-plane. Certain physical processes preserve the minimum phase property, and as such, the operators must arise mathematically as product-convolution operators of a very restrictive form. The basic mathematical model shows Q-attenuation arises as a simple consequence of the minimum phase preservation property for seismic signal propagation. In contrast to stationary filter processes, in Q-attenuation, not one but two data measurements are necessary for a complete determination of the attenuation characteristics. But only two.

This work is a summary of a sequence of papers on minimum phase properties.

INTRODUCTION

In geophysics applications there is a long standing assumption (see for instance Sherwood and Trorey (1965)) that in a horizontally stratified absorptive earth with vertically traveling plane compressional waves, transmitted waves are translates of minimum phase functions. There have been field experiments (see Ziolkowski and Bokhorst (1993)) supporting the assertion that, for instance, the source signature of a dynamite blast is itself minimum phase. This provides the conceptual basis for a mathematical inverse problem that we wish to consider, in identifying the specific linear operator that represents a specific instance of wave propagation through the earth. Supposing that these minimum phase waveforms are the output of a linear operator that appropriately encodes the material properties of the earth, how do we determine what the operator is? That is, what observations or recorded signals are required from a particular seismic experiment to determine the linear operator in question.

Some previous work of the present authors considered the case of modelling minimum phase preserving operators for use in discrete time digital signal processing. Real physics, however, occurs in continuous time, so it is appropriate to ask the analogous questions about such physical linear operators in a continuous time setting, which is what is done in the present work.

The paper is structured as follows. We first define what we mean by minimum phase, based on an analogy with the discrete time model. Some particular examples of analytic

*York University

minimum phase signals are given, which are useful both for theoretical work as well as numerical implementations. The structure of minimum phase preserving linear operators is then revealed: they turn out to be simply product-composition operators. An interpretation of these operators is obtained which reveals them to model the well-known Q-attenuation that arises in seismic wave propagation. Finally, we solve the identification problem, showing that two measurements of outputs from the linear operator suffice to fully specify any minimum phase preserving operator.

MINIMUM PHASE SIGNALS

Minimum phase, or more precisely minimum phase delay, is terminology borrowed from filtering theory in digital signal processing (see Oppenheim and Schafer (2010)), which refers to the property of certain filters to minimize the phase delay introduced by a filter with a specified amplitude response. Implementable, discrete-time filters are represented by rational functions in the z-transform domain (ratio of polynomials in z); they are minimum phase precisely when the roots and poles of the function lie outside the unit disk in the complex plane.

Minimum phase *signals*[†] in discrete time do not have the zero/pole characterization. However, they do have the physical property that energy is concentrated near the front of the signal; this is equivalent to saying their z-transform is an analytic outer function on the unit disk (see Hoffman (2007) or Helson (1995)).

In continuous time, we borrow this analogy and define a causal signal on the positive real line to be minimum phase if its Laplace transform (or s-transform) is an outer function on the right half-plane of the complex plane. That is, given a function $f \in L^2(\mathbb{R}^+)$, its s-transform is the function

$$F(s) = \int_0^\infty f(t)e^{-ts}\, dt, \tag{1}$$

which, by uniform convergence of the integral, defines an analytic function in complex variable s on the right half of the complex plane,

$$\{s \in \mathbb{C} : Re(s) > 0\}. \tag{2}$$

When $F(s)$ is an outer function, we say that the signal $f(t)$ is minimum phase.

The precise definition of outer is rather technical, but standard in complex analysis. The analytic function $F(s)$ is outer if its values on the right half plane are determined by its values on the imaginary axis, via the formula

$$F(s) = \lambda \exp\left(\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{ys+i}{y+is} \log|F(iy)| \frac{dy}{1+y^2}\right), \text{ for } Re(s) > 0. \tag{3}$$

Since $F(iy)$ is nothing more than the Fourier transform of the signal $f(t)$, this definition states that a signal is minimum phase if its s-transform is completely determined by the log

[†]in contrast to filters

amplitude (Fourier) spectrum of the signal. Equivalently, the amplitude and phase of the Fourier transform are related by the Kramer-Kronig relations.

Note in particular that the factor $\log|F(iy)|/(1+y^2)$ must be integrable, which places some constraints on the Fourier transform. This includes that the Fourier spectrum of a minimum phase signal must not have too many zeros (or else the logarithm causes a divergence to minus infinity), and the spectrum must not decay much faster than exponential as y tends to infinity (or else $\log|F(iy)|/(1+y^2)$ gets too negative too fast, and is not integrable).

While this definition is somewhat complicated and perhaps unmotivated, there are many mathematical results that make this a particularly powerful and useful concept (see Helson (1995)). One useful observation is that when the s-transform of a signal is a rational function in s (a ratio of polynomials in s), then the signal is minimum phase if and only if the zeros and poles of the s-transform $F(s)$ all lie outside the right half of the complex plane.

Another useful observation is that a function $F(s)$ is outer on the half plane if and only if its image function

$$G(z) = \frac{1}{1+z} F\left(\frac{1-z}{1+z}\right) \tag{4}$$

is an outer function in the unit disk (the z-transform of a discrete time, minimum phase signal). See Gibson and Lamoureux (2011b) for details.

The next section gives a few useful examples of continuous-time minimum-phase signals, using the rational function identification.

EXAMPLES OF MINIMUM PHASE SIGNALS

The canonical example of a minimum phase signal is the Dirac delta function, whose Laplace transform is the constant function one.

$$f(t) = \delta(t), \qquad F(s) = 1. \tag{5}$$

Such a function has, in principle, infinite energy and is not possible to create exactly in a real physical situation, nor in a numerical simulation.

More useful are finite energy signals which include an exponential decay. Following are some decaying polynomials, and their Laplace transforms:

$$f(t) = \exp(-t), \text{ for } t > 0, \qquad F(s) = \frac{1}{s+1}, \tag{6}$$

$$f(t) = t\exp(-t), \text{ for } t > 0, \qquad F(s) = \frac{1}{(s+1)^2}, \tag{7}$$

$$f(t) = t^2\exp(-t), \text{ for } t > 0, \qquad F(s) = \frac{1}{(s+1)^3}. \tag{8}$$

Each of these functions has transform with a pole at $s = -1$ which is in the left half plane, and thus are minimum phase.

Decaying sinusoids also give minimum phase signals, as indicated in these two examples with their corresponding Laplace transforms:

$$f(t) = \sin(t)\exp(-t), \text{ for } t > 0, \qquad F(s) = \frac{1}{(s+1)^2+1}, \tag{9}$$

$$f(t) = \cos(t)\exp(-t), \text{ for } t > 0, \qquad F(s) = \frac{s+1}{(s+1)^2+1}. \tag{10}$$

Here, the poles are in the left half plane at $s = -1 \pm i$; in the cosine case, there is also a zero at $s = -1$. More generally, a scaled sinusoid a specified angular frequency ω and decay parameter $a > 0$ has transform given by

$$f(t) = \sin(\omega t)\exp(-at), \text{ for } t > 0, \qquad F(s) = \frac{\omega}{(s+a)^2+\omega^2}, \tag{11}$$

$$f(t) = \cos(\omega t)\exp(-at), \text{ for } t > 0, \qquad F(s) = \frac{s+a}{(s+a)^2+\omega^2}. \tag{12}$$

Again, with poles at $s = -a \pm i\omega$ in the left half plane, the Laplace transform indicates these decaying sinusoids are minimum phase. Figure 1 shows some additional minimum phase sinusoids, with various levels of differentiability at initial time $t = 0$. Such sinusoids are useful for numerical codes that require smooth, minimum phase inputs.

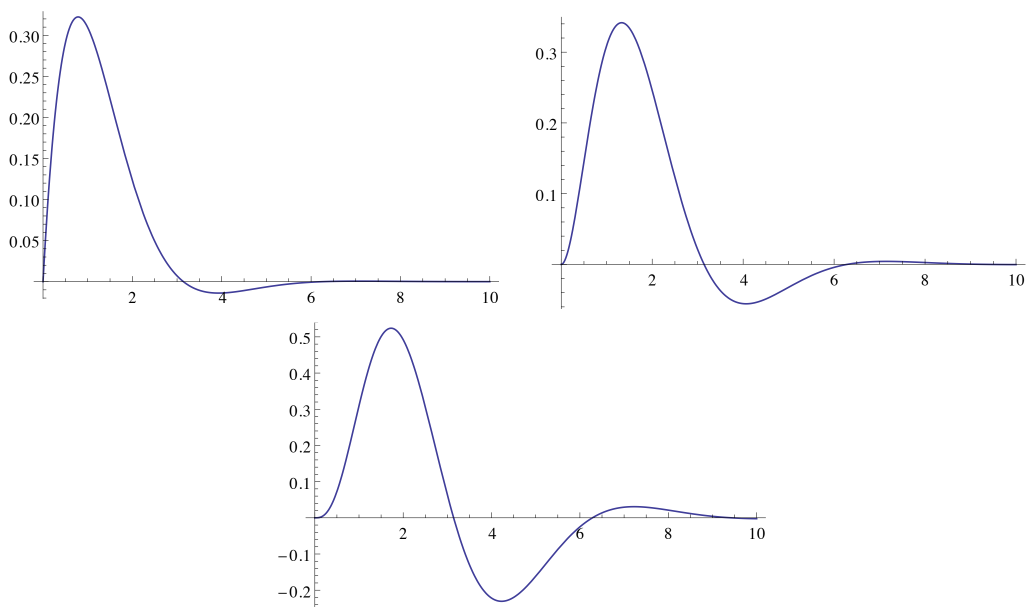

FIG. 1. Three analytic minimum phase signals: $\sin(t)\exp(-t)$, $t\sin(t)\exp(-t)$, $t^2\sin(t)\exp(-t)$.

The zero phase Rickert wavelet does not have a minimum phase counterpart. The problem is that the Gaussian $\exp(-t^2)$ has a Gaussian for its Fourier transform,

$$f(t) = \exp(-t^2), \text{ for all } t, \qquad \widehat{f}(\omega) = \exp(-\omega^2), \tag{13}$$

and thus its logarithm includes a quadratic term, which will not be integrable relative to the outer function weighting. The Rickert wavelet $(1-t^2)\exp(-t^2)$ will have a similar Gaussian factor in the Fourier domain, so the problem is still there.

A minimum phase approximation to the Rickert wavelet may be obtained by truncating the Fourier amplitude spectrum for large frequencies and adding a small ϵ-perturbation. However, for practical purposes, the analytic functions given above with decaying sinusoids may be more useful. They are certainly more precisely specified.

MINIMUM PHASE PRESERVING OPERATORS

It has been observed, and is generally assumed, that impulsive seismic sources such as a dynamite blast create minimum phase signals (see Sherwood and Trorey (1965)) and that such signals remain minimum phase as they propagate through the earth (see Ziolkowski and Bokhorst (1993)). In the mathematical modelling of a seismic experiment, is is important to include operators that preserve the minimum phase property of signals.

In discrete time, we have the following theorem (from Gibson et al. (2011)), which characterizes those operators that preserve (discrete) minimum phase functions and their offset versions (shifted outer functions):

Theorem 1 *Let $A : H^2(\mathbb{D}) \to H^2(\mathbb{D})$ be a bounded linear operator that preserves the set of shifted outer functions. Then A is a product-composition operator,*

$$A = M_\psi C_\phi, \tag{14}$$

where M_ψ is multiplication by shifted outer function ψ and C_ϕ is right composition with a shifted outer function ϕ that maps the unit disk to itself.

Also from Gibson et al. (2011), we have a result a continuous time version that identifies those operators which preserve outer functions defined on the right half of the complex plane, $\mathbb{C}_+$:

Theorem 2 *Let $A : H^2(\mathbb{C}_+) \to H^2(\mathbb{C}_+)$ be a bounded linear operator that preserves the set of outer functions on the half plane. Then A is a product-composition operator,*

$$A = M_\psi C_\phi, \tag{15}$$

where M_ψ is multiplication by an outer function ψ and C_ϕ is right composition with an outer function ϕ that maps the right half plane to itself.

Because of the isometry between Hardy spaces on the disk (corresponding to signals in discrete time) and Hardy spaces on the right half of the complex plane (corresponding to signals in continuous time), the result extends to the following:

Theorem 3 *Let $A : H^2(\mathbb{C}_+) \to H^2(\mathbb{C}_+)$ be a bounded linear operator that preserves the set of shifted outer functions. Then A is a product-composition operator,*

$$A = M_\psi C_\phi, \tag{16}$$

where M_ψ is multiplication by shifted outer function ψ and C_ϕ is right composition with a shifted outer function ϕ that maps the right half plane to itself.

For continuous time, a shifted outer function is in the form

$$F(s) = e^{-\tau s} F_o(s), \tag{17}$$

with outer function $F_o(s)$, shifted by time parameter $\tau \geq 0$. These are s-transforms of time-shifted minimum phase signals,

$$f_\tau(t) = f_o(t - \tau), \text{ for } t \geq \tau. \tag{18}$$

There are some details in Theorem 3 that require further investigation. The discrete integer-time shifts in sampled time do not map easily to the continuous time shifts observed in Theorem 3. There is also the question of identifying minimum phase in continuous time with the energy front-loading expected in physical impulsive signals. Part of the difficulty comes from the Pauli problem which notes the ambiguity in discriminating between functions in continuous time that have the same amplitude in time, and (Fourier) amplitude in frequency (see Ismagilov (1996)). This is a work in progress, and we plunge along as necessary.

Q-ATTENUATION

The composition portion of the linear operator $A = M_\psi C_\phi$ gives the physical process of frequency-dependent exponential decay of a signal, which is usually identified as Q-attenuation. To see this, set $A = C_\phi$, a simple composition operator. Such a linear map transforms a function $F(s)$ in the s-transform space to a new function $F(\phi(s))$. Thus, a given signal $f(t)$ with s-transform

$$F(s) = \int_0^\infty f(t) e^{-ts}\, dt \tag{19}$$

is mapped to a new function

$$F(\phi(s)) = \int_0^\infty f(t) e^{-t\phi(s)}\, dt. \tag{20}$$

Thus, locally, function $f(t)$ is transformed to $f(t)e^{-t\phi(s)}$, which is a frequency-dependent exponential decay that increases with time.

More particularly, with $s = iy$ on the imaginary axis, at frequency y, the function $f(t)$ is locally transformed to

$$f(t) e^{-t\phi(iy)}. \tag{21}$$

We can identify the real part of the exponent $-t\phi(iy)$ as $-\pi t y / Q(y)$, to give the frequency-dependent Q parameter.

Note that by analyticity, the real and imaginary parts of $\phi(iy)$ are Hilbert transform pairs, so the Q-attenuation expressed in this form accounts for appropriate phase delays to ensure the minimum phase preserving property.

The multiplication portion of the linear operator $A = M_\psi C_\phi$ gives the physical effect of a stationary-in-time filtering process. This is easily seen from the observation that a multiplication in the s-tranform domain corresponds to a convolution in the time domain, which is precisely stationary filtering.

IDENTIFICATION OF MINIMUM PHASE PRESERVING OPERATORS

A minimum phase preserving operator is necessarily a product-composition operator,

$$A = M_\psi C_\phi. \tag{22}$$

Recovering the two functions ψ, ϕ will specify the operator entirely.

Begin with a causal, exponentially decaying signal $f(t) = e^{-t}$, for $t \geq 0$. Its Laplace transform is $F(s) = \frac{1}{s+1}$. Its image under the linear map A is

$$(AF)(s) = \psi(s)\frac{1}{\phi(s)+1}. \tag{23}$$

Next, take the causal signal $g(t) = te^{-t}$, for $t \geq 0$. Its Laplace transform is $G(s) = \frac{1}{(s+1)^2}$. Its image under the linear map A is

$$(AG)(s) = \psi(s)\frac{1}{(\phi(s)+1)^2}. \tag{24}$$

The ratio of the two image functions is

$$\frac{AF}{AG}(s) = \phi(s) + 1, \tag{25}$$

so this one ratio recovers function ϕ. The product

$$(AF)(s)\frac{AF}{AG}(s) = \psi(s) \tag{26}$$

recovers the other function ψ.

Thus, by measuring the output of operator A on the two causal signals

$$f(t) = e^{-t}, \qquad g(t) = te^{-t}, \tag{27}$$

we obtain sufficient information to completely recover functions ϕ, ψ which completely specify operator $A = M_\psi C_\phi$.

In the stationary case, $A = M_\psi$, of course only one single measurement is required.

CONCLUSIONS

We have shown the outline of an argument that suggests previous work on minimum phase preserving linear operators in discrete time can be extended to the continuous time case. In this situation, the geophysical problem of modelling the propagation of seismic energy through the earth is represented by linear operators which preserves the minimum phase property of signals. Such operators are necessarily product-composition operators, and as such, are uniquely determined by two analytic functions. The first function, ψ, represent a stationary filter, and the second, ϕ, represents a frequency dependent Q-attenuation. Such an operator can be uniquely characterized by its action on two specific causal signals, the decaying exponential $\exp(-t)$ and its polynomial counterpart $t\exp(-t)$.

ACKNOWLEDGEMENTS

The authors gratefully acknowledge the continuing support of the sponsors of the CREWES and POTSI research consortia, and our funding partners PIMS, NSERC, Mitacs and Mprime.

REFERENCES

Gibson, P. C., and Lamoureux, M. P., 2011a, Constructive solutions to Polya-Schur problems: arXiv, **1108.0043**.

Gibson, P. C., and Lamoureux, M. P., 2011b, Identification of minimum phase preserving operators on the half line: arXiv, **1109.2212**.

Gibson, P. C., Lamoureux, M. P., and Margrave, G. F., 2011, Outer preserving linear operators: Journal of Functional Analysis, **261**, 2656–2668.

Helson, H., 1995, Harmonic analysis: Henry Helson.

Hoffman, K., 2007, Banach spaces of analytic functions: Dover.

Ismagilov, R. S., 1996, On the Pauli problem: Journal of Functional Analysis, **30**, No. 2, 138–140.

Lamoureux, M. P., and Margrave, G. F., 2007, An analytic approach to minimum phase signals, Research Report 19, CREWES, http://www.crewes.org.

Martínez-Avendano, R. A., and Rosenthal, P., 2007, An introduction to operators on the Hardy-Hilbert space, vol. 237 of *Graduate texts in mathematics*: Springer.

Oppenheim, A. V., and Schafer, R. W., 2010, Discrete time signal processing, Signal processing series: Prentice-Hall, third edn.

Sherwood, J. W. C., and Trorey, A. W., 1965, Minimum phase and related properties of the reponse of a horizontally stratified absorptive earth to phane acoustic waves: Geophysics, **30**, No. 2, 191–197.

Ziolkowski, A., and Bokhorst, K., 1993, Determination of the signature of a dynamic source using source scaling, part 2: Experiment: Geophysics, **58**, No. 8, 1183–1194.