Can we see a velocity ramp, in reflection seismic data

Michael P. Lamoureux, Peter C. Gibson, Gary F. Margrave

ABSTRACT

We answer the question: what happens when a seismic wave propagates from one region of constant velocity to another region of different velocity, through a smooth transition zone – and can we detect this with seismic reflection data?

This work presents an exact analytic solution for the case of a linear ramp velocity in the transition zone, and demonstrates that for long wavelengths, the ramp looks essentially like a jump discontinuity in the medium, with the corresponding reflection and transmission coefficients. For short wavelengths, the ramp provides essentially 100% transmission and no reflection.

We describe the mathematical details of this frequency-dependent reflection and transmission, and derive an estimate of the feature size of the transition zone that should be detectable. If we can detect zero crossing in the middle of the frequency range of our seismic measurements, for instance at 50Hz, we expect to be able to detect transition zones with length on the order of $L = c/100s^{-1}$, where c is the velocity of propagation.

A careful consideration is given to the two cases of varying the velocity parameter, one via variations in the density of the propagation medium, the other in varying the modulus of elasticity. The results are different, in particular there is a sign difference in the reflection coefficient, and a large amplitude difference in the transmission coefficient.

We also present the numerical result for the transmission and reflection of a delta spike through the velocity ramp, and observe the reflection is a broadened "boxcar" response, whose width is directly related to the width of the transition zone. The transmission of the spike, however, also results in a spike.

INTRODUCTION

Reflection seismic imaging is based on the principle that seismic waves are reflected from discontinuities in the elastic impedance of various rock layers and structures within the earth's subsurface. This reflected wave energy can be measured with physical instruments on the surface of the earth, and an image of the subsurface can be derived from the discontinuities. Discontinuity, however, can be a matter of degree: a zone within the earth where the impedance changes rapidly, but continuously, might appear like a sharp discontinuity to a slowly changing, or low frequency, seismic wave. Reflection is also a matter of degree: a transition zone, even if it is not very sharp, might generate reflections that are in principle detectable at the surface.

As we move into imaging based on full waveform inversion, it is important to understand the physical effects that such a transition zone might have on our seismic sensing instruments. In particular, we would like to know what types of transition zones (how big the zone, how rapid the transition) are detectable using our current instruments, Our approach in the paper is to start with a simple physical model of a velocity field with a smooth transition zone, build a mathematical model of wave propagation through this zone, and characterize both the reflection and transmission properties of the propagation. Here, we build analytic solutions that are exact answers for these specific physical models, and compute exact formulas for the reflection and transmission data that we can expect to receive. Of course, in seismic imaging, we are intensely interested in the numerical solution of the wave equation representing the propagation of seismic energy through complex geological structures. Exact analytic solutions are hard to come by in such situations, so simpler models will have to suffice.

Exact solutions to the one dimension wave equation are often possible; these can be used to construct more general solutions in higher dimensions, but also have direct utility of their own. In this short paper, we construct an exact solution for a wave propagating through a linearly varying velocity field, and use it to answer specific physical questions.

Four questions are resolved. First of all, the reflection and transmission coefficients of a monochromatic wave impinging on the interface should depend on frequency: we compute an exact answer. We verify that in the limiting case of large wavelengths, the ramp looks effectively like a jump discontinuity, with the corresponding reflection and transmission coefficients for this discontinuous case. For very short wavelengths, we verify that there is effectively 100% transmission of energy, and zero reflection.

Second, we examine what is the effect of varying the density parameter in the wave equation, versus varying the modulus of elasticity. The solutions are different, and can be seen in the computation of the reflection and transmission coefficients. In both cases, a conservation of energy law is observed across the transition zone.

Third, we show the connection between frequency sensitivity of our seismic experiment, and the types of transition zones we can detect. Essentially, the lower the frequencies that can be accurately recorded, the longer the transition zone (or more slowly changing velocities) that can be detected. Numerical estimates make this qualitative statement precise.

Finally, we show the result of propagating a delta spike through the velocity ramp. Numerically, we see the transmitted signal is still a spike, while the reflected signal broadens into a boxcar-like pulse. The width of the boxcar (in time) is directly related to the time of propagation across the transition zone. This can also be observed in the frequency domain as a sinc function representing the reflected frequency response of the linear velocity ramp.

We consider two realistic physical examples, include the case of a rather rapid change in velocity (a rapid change in a zone of length 100 meters) and the more gentle transition typically seen in a sedimentary basin.

VELOCITY MODEL AND LOCAL SOLUTIONS

The one dimensional "elastic" wave equation for a displacement field $u(x, t)$, of a disturbance travelling along a weighted string under tension, with density ρ and modulus of elasticity (bulk modulus) K is given in the standard form

$$\rho(x)\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(K(x)\frac{\partial u}{\partial x}\right). \tag{1}$$

We use this form as the 1D model, since this is the model directly applicable to elastic waves in three dimensions, with displacement $u(x,t)$ measured by geophones. For clarity, though, we note that the acoustic wave equation appears in a similar form,

$$\frac{1}{K(x)}\frac{\partial^2 p}{\partial t^2} = \frac{\partial}{\partial x}\left(\frac{1}{\rho(x)}\frac{\partial p}{\partial x}\right). \tag{2}$$

where here $p(x,t)$ represents acoustic pressure (which could be measured directly by hydrophones), ρ is density of the medium, and $K(x)$ comes from the equation of state. In the case of constant coefficients, these two equations are equivalent.

It is important to note that in the elastic equation (1), it is the modulus of elasticity that appears inside the derivative, while in the acoustic equation (2), it is the density that appears inside the derivative. For non-constant ρ, K, the position of these coefficients within the derivative has important physical consequences – in particular on the sign of reflection coefficients. However, throughout this article, we will only work with equation (1). The ratio $K/\rho = c^2$ is identified as the velocity of propagation, squared.

We model a situation where a wave propagates from a region of constant velocity c_1 to second region of constant velocity c_2, through a smooth transition zone of length L. For simplicity, we specify a linear velocity ramp $c(x) = c_1 + mx$ in the transition zone, $0 < x < L$, where $m = (c_2 - c_1)/L$ is the slope of the velocity ramp. With such a ramp, exact solutions can be obtained. We are primarily interested in determining what changes in velocity, c_1 to c_2 and lengths of transition zones L can be detected using usual seismic measurements for which exact analytic solutions can be obtained. A sample velocity field is shown in Figure 1.

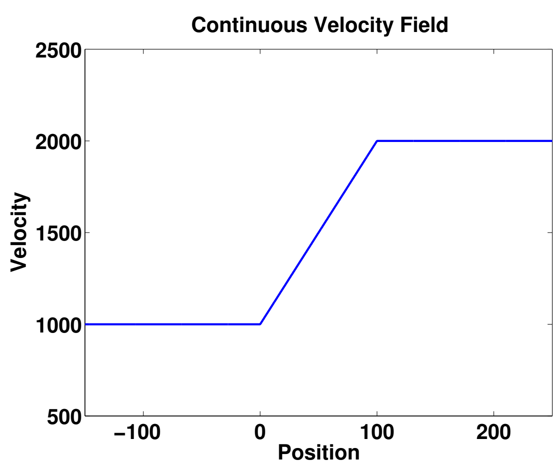

FIG. 1. Velocity field varying from $c_1 = 1000m/s$ to $c_2 = 2000m/s$, with a linear ramp over 100m.

Setting density $\rho(x) = c^{-2}(x)$ and constant modulus $K = 1$, we obtain a first case of the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2(x)\frac{\partial^2 u}{\partial x^2}, \tag{3}$$

solved via separation of variables, with $u(x,t) = X(x)T(t)$ to obtain basic oscillatory solutions

$$\begin{aligned} u(x,t) &= e^{i\omega(\pm x/c_1 - t)} && x < 0, && (4)\\ &= (c_1 + mx)^{1/2 \pm \sqrt{1/4-(\omega/m)^2}} e^{-i\omega t} && 0 < x < L, && (5)\\ &= e^{i\omega(\pm x/c_2 - t)} && x > L. && (6) \end{aligned}$$

We would then take linear combinations of these basic solutions, and apply boundary conditions at the interfaces to obtain unique solutions in a given physical setup.

It is convenient to express the middle solution in the form

$$u(x,t) = c(x)^n e^{-i\omega t}, \tag{7}$$

for exponents $n = 1/2 \pm \sqrt{1/4 - (\omega/m)^2}$. We call this the varying density case. Figure 2 shows a typical oscillatory wave moving through the transition zone. Notice that in the transition from left to right in FIgure 2, the spatial wavelength is increasing, as is the amplitude.

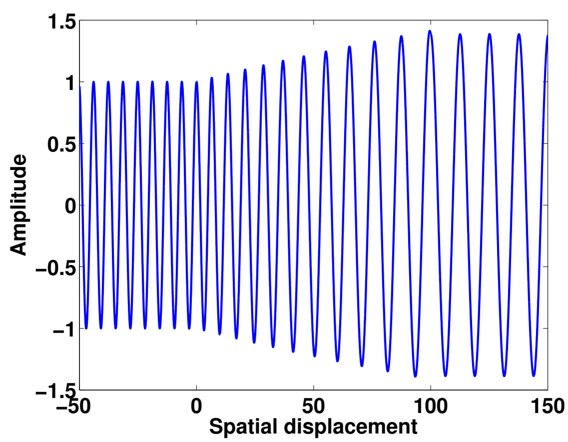

FIG. 2. Waveform moving through the transition zone $0 < x < 100$, varying density case.

A second solvable case of the wave equation is obtained by taking density $\rho = 1$ constant, and varying modulus in the form $K(x) = c^2(x)$, yielding the equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(c^2(x)\frac{\partial u}{\partial x}\right) \tag{8}$$

which has solutions of a similar form,

$$\begin{aligned} u(x,t) &= e^{i\omega(\pm x/c_1 - t)} && x < 0, && (9)\\ &= c(x)^n e^{-i\omega t} && 0 < x < L, && (10)\\ &= e^{i\omega(\pm x/c_2 - t)} && x > L. && (11) \end{aligned}$$

with the change that here, the exponents are $n = -1/2 \pm \sqrt{1/4 - (\omega/m)^2}$. We call this the varying modulus case. Figure 3 shows a typical wave in the transition zone. Notice that from left to right, the spatial wavelength is again increasing, but the amplitude is now decreasing, in contrast to the varying density case.

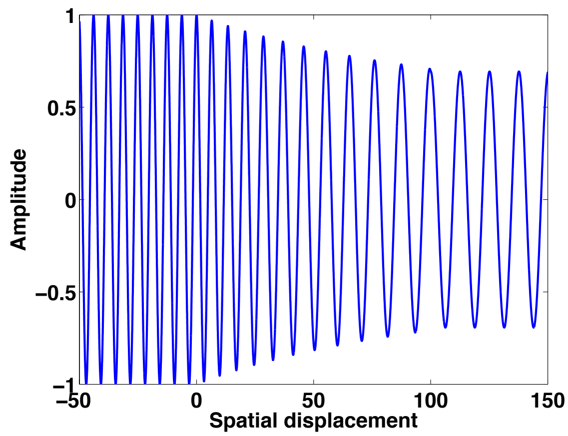

FIG. 3. Waveform moving through the transition zone $0 < x < 100$, varying modulus case.

REFLECTION AND TRANSMISSION COEFFICIENT - VELOCITY JUMP

As an illustrative example, we consider the case of a reflection and transmission through a simple velocity jump, of the form

$$\begin{aligned} c(x) &= c_1 \qquad x < 0, & (12) \\ &= c_2 \qquad x > 0. & (13) \end{aligned}$$

Solutions are of the form

$$\begin{aligned} u(x,t) &= e^{i\omega(\pm x/c_1 - t)} \qquad x < 0, & (14) \\ &= e^{i\omega(\pm x/c_2 - t)} \qquad x > 0. & (15) \end{aligned}$$

To compute reflection and transmission coefficients, set an incoming wave on the left of the form $e^{i\omega(x/c_1-t)}$, and hypothesize a reflected wave of the form $Re^{i\omega(-x/c_1-t)}$ and transmitted wave on the right of the form $Te^{i\omega(x/c_2-t)}$.

The initial wave and the reflected wave on the left sum together to give

$$u_{left} = e^{i\omega(x/c_1-t)} + Re^{i\omega(-x/c_1-t)} \tag{16}$$

The transmitted wave stands alone, giving

$$u_{right} = Te^{i\omega(x/c_2-t)}. \tag{17}$$

To obtain the coefficients R, T, we impose continuity on displacement u across the interface $x = 0$,

$$u_{left} = u_{right} \text{ at } x = 0 \tag{18}$$

and continuity of force

$$K_{left}(\partial_x u)_{left} = K_{right}(\partial_x u)_{right} \text{ at } x = 0, \tag{19}$$

yielding

$$1 + R = T \tag{20}$$

$$\frac{K_{left}}{c_1}(i\omega - i\omega R) = \frac{K_{right}}{c_2} i\omega T. \tag{21}$$

There are two cases to consider. First, when the modulus K is constant, and $\omega \neq 0$, the system reduces to

$$1 + R = T \tag{22}$$

$$1 - R = \frac{c_1}{c_2} T, \tag{23}$$

which has solution $R = (c_2 - c_1)/(c_2 + c_1), T = 2c_2/(c_2 + c_1)$.

In the second case, with varying modulus ($K_{left} = c_1^2, K_{right} = c_2^2$), the system reduces to

$$1 + R = T \tag{24}$$

$$1 - R = \frac{c_2}{c_1} T, \tag{25}$$

which has solution $R = -(c_2 - c_1)/(c_2 + c_1), T = 2c_1/(c_2 + c_1)$. The negative reflection coefficient indicates that there is a flip in polarity upon reflection.

Observe that in both cases, we have a conservation of energy constraint, namely

$$\frac{K_{left}}{c_1} R^2 + \frac{K_{right}}{c_2} T^2 = \frac{K_{left}}{c_1}(1)^2, \tag{26}$$

where of course the $(1)^2$ represents the energy of the initial wave impinging on the interface.

REFLECTION AND TRANSMISSION COEFFICIENTS - VELOCITY RAMP, VARYING DENSITY

To compute reflection and transmission coefficients across a velocity ramp, set an incoming wave on the left of the form $e^{i\omega(x/c_1 - t)}$, and hypothesize a reflected wave of the form $Re^{i\omega(-x/c_1 - t)}$ and transmitted wave on the right of the form $Te^{i\omega(x/c_2 - t)}$. In the transition region, set the wave to a linear combination of the solutions $c(x)^{1/2 \pm \sqrt{1/4 - (\omega/m)^2}} e^{-i\omega t}$. This gives three regional solutions,

$$u_{left} = e^{i\omega(x/c_1 - t)} + Re^{i\omega(-x/c_1 - t)}, \tag{27}$$

$$u_{trans} = Ac(x)^{n_1} e^{-i\omega t} + Bc(x)^{n_2} e^{-i\omega t}, \tag{28}$$

$$u_{right} = Te^{i\omega(x/c_2 - t)}, \tag{29}$$

with $n_1 = 1/2 + \sqrt{1/4 - (\omega/m)^2}, n_2 = 1/2 - \sqrt{1/4 - (\omega/m)^2}$. This corresponds to the varying density case identified in Section 2.

To find the four coefficients R, T, A, B, set four continuity conditions

$$
\begin{aligned}
u_{left} &= u_{trans} \text{ at } x = 0, && (30)\\
u_{trans} &= u_{right} \text{ at } x = L, && (31)\\
K_{left}\left(\partial_x u_{left}\right) &= K_{trans}\left(\partial_x u_{trans}\right) \text{ at } x = 0, && (32)\\
K_{trans}\left(\partial_x u_{trans}\right) &= K_{right}\left(\partial_x u_{right}\right) \text{ at } x = L. && (33)
\end{aligned}
$$

This yields the four equations

$$
\begin{aligned}
1 + R &= c_1^{n_1} A + c_1^{n_2} B, && (34)\\
c_2^{n_1} A + c_2^{n_2} B &= e^{i\omega L/c_2} T, && (35)\\
\frac{i\omega}{c_1} - \frac{i\omega}{c_1} R &= m n_1 c_1^{n_1-1} A + m n_2 c_1^{n_2-1} B, && (36)\\
m n_1 c_2^{n_1-1} A + m n_2 c_2^{n_2-1} B &= \frac{i\omega}{c_2} e^{i\omega L/c_2} T. && (37)
\end{aligned}
$$

Setting $r_1 = (c_2/c_1)^{n_1}, r_2 = (c_2/c_1)^{n_2}$ and $k_1 = mn_1/i\omega, k_2 = mn_2/i\omega$, these equations are written in matrix form as

$$
\begin{bmatrix} 1 & 1 & -1 & 0 \\ r_1 & r_2 & 0 & -1 \\ k_1 & k_2 & 1 & 0 \\ k_1 r_1 & k_2 r_2 & 0 & -1 \end{bmatrix}
\begin{bmatrix} c_1^{n_1} A \\ c_1^{n_2} B \\ R \\ e^{i\omega L/c_2} T \end{bmatrix}
=
\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}. \tag{38}
$$

Invert the matrix $\mathbf{M}$ in Equation 38 to obtain the solution, and note that the coefficients A, B, R, T depend on frequency ω:

$$
\begin{bmatrix} c_1^{n_1} A \\ c_1^{n_1} B \\ R \\ e^{i\omega L/c_2} T \end{bmatrix}
= \mathbf{M}^{-1}
\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}. \tag{39}
$$

Figure 4 shows a plot of the reflection and transmission coefficients, as well as a check on conservation of energy, for a sample case where the velocity ratio is $c_2/c_1 = 2$ and a moderate length transition zone with slope $m = 10$. This corresponds to the velocity model in Figure 1, where the transition zone has length 100m. The plot verifies that

$$
|R(\omega)|^2 + \frac{1}{2}|T(\omega)|^2 = 1. \tag{40}
$$

Observe that for frequency $\omega \approx 0$, we obtain

$$
|R(\omega)| \approx \frac{1}{3} = \frac{c_2 - c_1}{c_2 + c_1}, \qquad |T(\omega)| \approx \frac{4}{3} = \frac{2c_2}{c_2 + c_1}, \tag{41}
$$

which agrees with the solution for a simple jump, as discussed in the last section. Physically, this says for low frequency sources (long wavelengths), the ramped velocity change looks a lot like a simple jump.

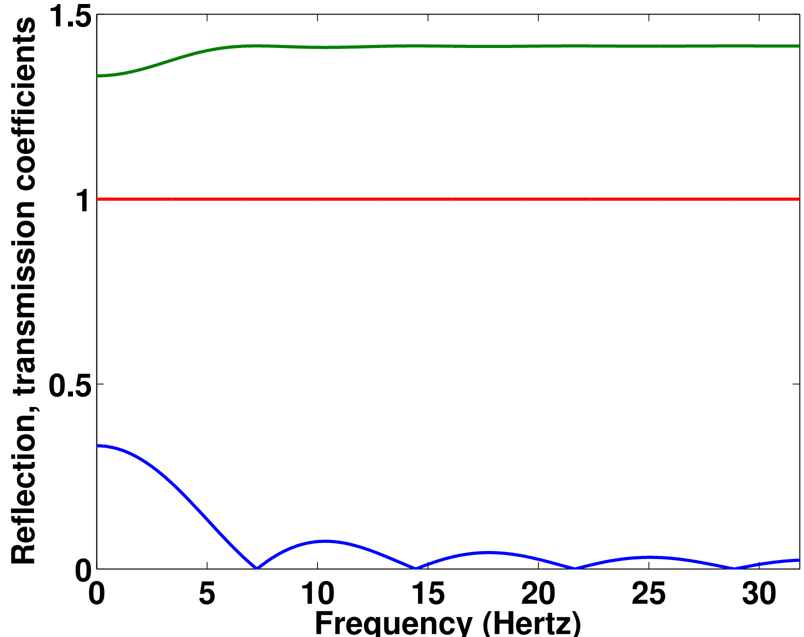

FIG. 4. Frequency dependent coefficients: reflection (lower), transmission (upper), and energy check (middle).

We also observe that for frequency ω large, we obtain

$$|R(\omega)| \approx 0, \qquad \frac{1}{2}|T(\omega)| \approx 1, \tag{42}$$

which says that there is very little energy in the reflected signal, and lots of energy in the transmitted signal. That is to say, high frequency sources (short wavelength) pass through the ramp with very little change in energy, although the amplitude changes by a factor of $\sqrt{2}$.

Also notice that at about 7 Hertz, there is a notch (zero) in the reflection coefficient, which is replicated at higher frequencies. This resonant phenomena should be detectable in the seismic signal if there is sufficient resolution at low frequencies.

For a longer transition zone with a more modest slope of, say $1/s$, more typical of the gradual linear increase in velocity seen in a sedimentary basin, this zero in the reflection response moves to lower frequencies (0.7Hz), as shown in Figure 5. This suggests seismic data in much lower frequency ranges would be required to see this gradual velocity increase in the seismic recording.

REFLECTION AND TRANSMISSION COEFFICIENTS - VELOCITY RAMP, VARYING MODULUS

We can solve for reflection and transmission coefficients for the second variant of the wave equation,

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(c^2(x)\frac{\partial u}{\partial x}\right), \tag{43}$$

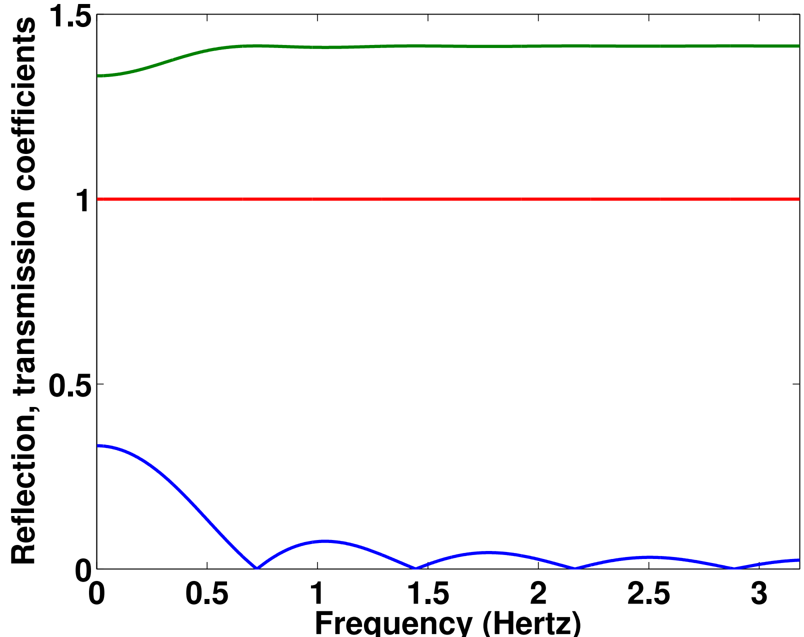

FIG. 5. Frequency dependent coefficients: reflection (lower), transmission (upper), and energy check (middle).

using the same system of equations, except change coefficients n_1, n_2 to

$$n_1 = -1/2 + \sqrt{1/4 - (\omega/m)^2}, \qquad n_2 = -1/2 - \sqrt{1/4 - (\omega/m)^2}. \tag{44}$$

Notice that the modulus of elasticity $K(x) = c^2(x)$ is continuous across all interfaces, so it can be eliminated from the system of equations 30–33.

Figure 6 shows a plot of the reflection and transmission coefficients, as well as a check on the conservation of energy, with

$$|R(\omega)|^2 + 2|T(\omega)|^2 = 1, \tag{45}$$

where the coefficient $2 = K/c$ comes from the right hand side of the wave field.

We observe the reflection coefficients look very similar to the previous case. The transmission coefficient differs by a factor of two.

In particular, for ω small, we obtain a value of $T(\omega) \approx 2/3$, which agrees with the jump case (Section titled "Velocity Jump") using a varying bulk modulus. For large ω, there is almost no reflected energy, and the transmitted signal carries almost all the energy, while its amplitude is reduced by a factor of $1/\sqrt{2}$. And again, there is a a notch in the reflection response at about 7 Hz.

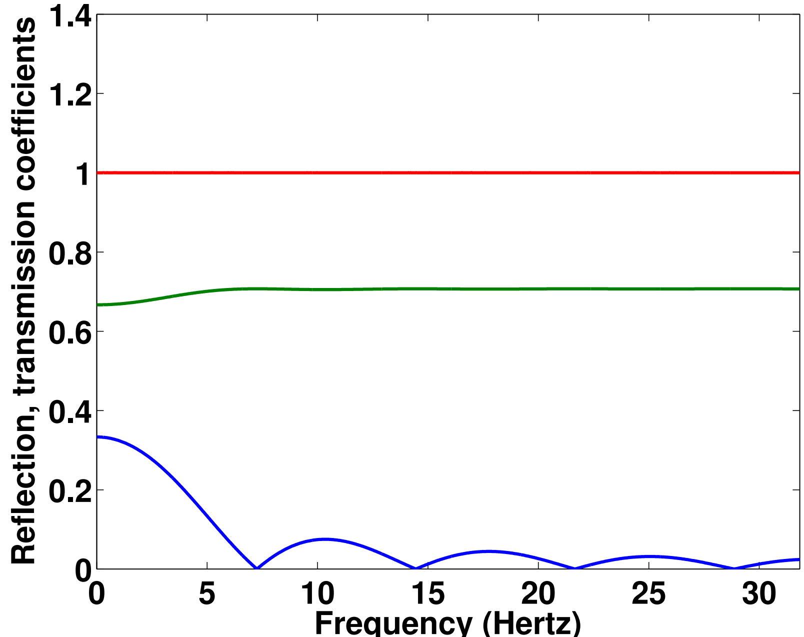

FIG. 6. Frequency dependent coefficients: reflection (lower), transmission (middle), and energy check (top).

ANALYTIC SOLUTION TO REFLECTION COEFFICIENT

Using Cramer's rule, and a little help from Mathematica™, one can obtain an exact solution for the reflection coefficient

$$R(\omega) = \frac{r_1 - k_1 r_1 - k_2 r_1 + k_1 k_2 r_1 - r_2 - k_1 r_2 + k_2 r_2 - k_1 k_2 r_2}{r_1 - k_1 r_1 + k_2 r_1 + k_1 k_2 r_1 - r_2 - k_1 r_2 + k_2 r_2 + k_1 k_2 r_2} \tag{46}$$

which simplifies to

$$R(\omega) = \frac{(r_2 - r_1)(n_1 + n_2)}{2(r_1 - r_2)(i\omega/m) + 2(r_1 + r_2)\sqrt{1/4 - (\omega/m)^2)}}. \tag{47}$$

In the varying density case, we have $n_1 + n_2 = 1$, and setting $a = \sqrt{1/4 - (\omega/m)^2}$, we obtain

$$R(\omega) = \frac{(c_2/c_1)^a - (c_2/c_1)^{-a}}{2(i\omega/m)\left[(c_2/c_1)^a - (c_2/c_1)^{-a}\right] + 2a\left[(c_2/c_1)^a + (c_2/c_1)^{-a}\right]}. \tag{48}$$

Note that $R(0) = (c_2 - c_1)/(c_2 + c_1)$, so there is a positive reflection in this varying density case, at low frequencies.

In the varying modulus case, we have $n_1 + n_2 = -1$, and we get exactly the negative of this previous solution,

$$R(\omega) = -\frac{(c_2/c_1)^a - (c_2/c_1)^{-a}}{2(i\omega/m)\left[(c_2/c_1)^a - (c_2/c_1)^{-a}\right] + 2a\left[(c_2/c_1)^a + (c_2/c_1)^{-a}\right]}. \tag{49}$$

Here, we find $R(0) = -(c_2 - c_1)/(c_2 + c_1)$, so there is a negative reflection in this varying modulus case, at low frequencies.

This is not surprising: it is well known that the reflection coefficients in the wave equation changes sign depending on whether one has a change in density, or a change in bulk modulus. The above calculation confirms this is true for each frequency reflecting off a velocity ramp.

A similar procedure will give the analytic solution for the transmission coefficients.

DETECTING THE RAMP

The plots of the reflection frequency response in Figures 4, 5, and 6 are suggestive of a sinc function. For ω/m large, we have the approximation

$$a = \sqrt{1/2 - (\omega/m)^2} \approx i\omega/m, \tag{50}$$

which yields an approximation for the reflectivity response in Equation (48) as

$$\begin{aligned} R(\omega) &\approx \frac{(c_2/c_1)^{i\omega/m} - (c_2/c_1)^{-i\omega/m}}{4(i\omega/m)(c_2/c_1)^{i\omega/m}} && (51) \\ &\approx \operatorname{sinc}(\frac{\omega}{m}\log\frac{c_2}{c_1})\frac{\log\frac{c_2}{c_1}}{2}(c_2/c_1)^{-i\omega/m}. && (52) \end{aligned}$$

Ignoring the phase shift $(c_2/c_1)^{-i\omega/m}$, we see the reflectivity is approximately a sinc function, scaled by a factor $\frac{1}{2m}\log\frac{c_2}{c_1}$.

To detect the ramp, it is reasonable to require that the first zero of the sinc function occur somewhere in the seismic band – from this data we can make an estimate of the shape of the sinc function. The first zero occurs at

$$\frac{\omega}{m}\log\frac{c_2}{c_1} = \pi, \tag{53}$$

or equivalently in frequency $f = \omega/2\pi$ as

$$f = \frac{m}{2\log\frac{c_2}{c_1}}. \tag{54}$$

In the model for Figure 4, we have a velocity ratio c_2/c_1 of 2, and a slope of $m = 10s^{-1}$, which puts the first notch at frequency

$$f = \frac{10}{2\log 2} = 7.21Hz. \tag{55}$$

in good agreement with the notch displayed in the figure. In Figure 5, the notch moves to

$$f = \frac{1}{2\log 2} = 0.721Hz, \tag{56}$$

as expected.

It is also useful to express the notch frequency in terms of the length of the transition zone. With $m = (c_2 - c_1)/L$ and $\log \frac{c_2}{c_1} \approx c_2/c_1 - 1$ this gives the first zero of the sinc function appearing at

$$f \approx \frac{(c_2 - c_1)/L}{2(c_2/c_1 - 1)} = \frac{c_1}{2L}. \tag{57}$$

With f set in the middle of the seismic band, say perhaps $f = 50Hz$, we get an estimate of a detectable transition zone of length L with

$$L \approx \frac{c_1}{2 \cdot 50} = \frac{c_1}{100}. \tag{58}$$

So, for instance, with an initial wave speed velocity of $c_1 = 1000m/s$, we can expect to detect a transition zone of length $L \approx 10m$. A shorter transition of length $L \approx 1m$ would put the first zero of the sinc function at about 500Hz, which is likely outside what we can detect reliably with current technology. In such a case, the transition zone effectively appears as a jump in reflectivity. A much longer transition of length $L \approx 100m$ puts the zero at 5Hz, which would require some careful low frequency processing to detect.

VARYING DENSITY AND BULK MODULUS

One can carry out the same analysis where both density and modulus are varying, in the form

$$\rho(x) = c(x)^{\alpha-2} \tag{59}$$
$$K(x) = c(x)^{\alpha} \tag{60}$$

for some fixed α, with x in the interval $[0, L]$, and extend by continuity to constants on the rest of the real line. Solutions in the transition region are of the form

$$u(x,t) = c(x)^n e^{-i\omega t}, \tag{61}$$

where $n = (1-\alpha)/2 \pm \sqrt{(1-\alpha)^2/4 - (\omega/m)^2}$. We skip the details – the same matrix inversion is required.

The case $\alpha = 1$ would be particularly interesting, for in this situation a monochromatic waveform in the transition zone would maintain a constant amplitude, in contrast to the increase (Figure 2) or decrease (Figure 3) in amplitude shown earlier. This interesting case is not discussed further here.

TRANSMISSION, REFLECTION OF A DELTA SPIKE

It is interesting to ask how an arbitrary waveform impinging on the velocity ramp is transformed into two resulting waveforms, the reflected waveform and the transmitted waveform. In principle, this can be computed by convolving the initial waveform with the filter response of the ramp, for both reflection and transmission.

The filter response is simply the result of initiating a delta spike on the left of the velocity ramp, and allowing it to travel into the ramp, creating a reflected and a transmitted waveform.

Conveniently, we have already computed the filter response for the ramp in the frequency domain, as Equations 46 and 47 represent analytic solution for reflection. Taking the inverse Fourier transform will give the filter response in space.

However, computing the exact inverse Fourier transform by hand is beyond what we are able to do in this short paper. As an alternative, we can simply sample in the frequency domain, then numerically compute the inverse Fast Fourier Transform. Again working with our velocity model from Figure 1, we consider the case of a transition zone of 100m, where the velocity changes from 1000 m/s to 2000 m/s. Figure 7 shows the resulting reflection for a impulsive source, which is the impulse response of the ramp in reflecting a delta spike. Observe that exactly one reflection is obtained, but it is much wider than a delta spike, about 0.15 seconds long.

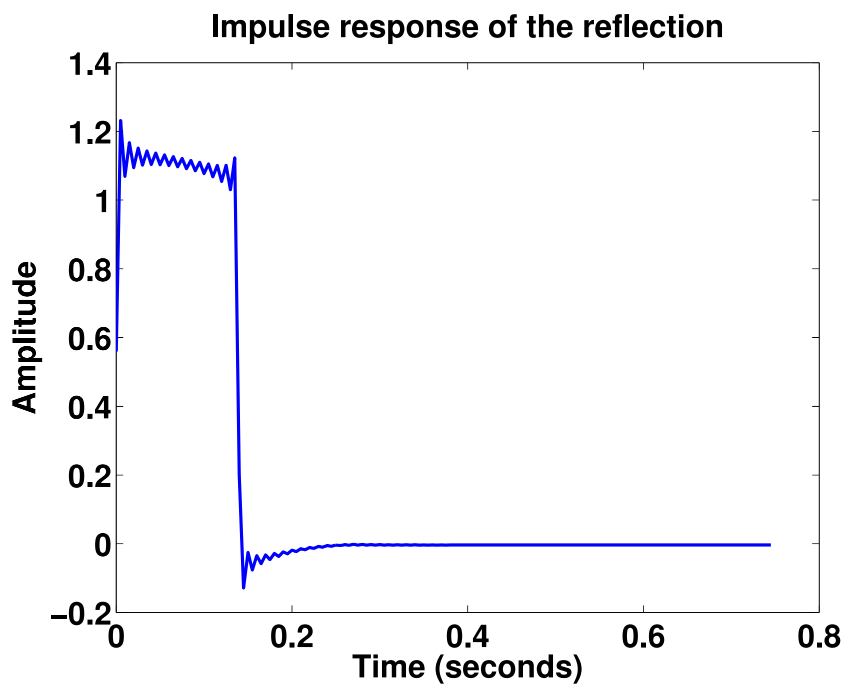

FIG. 7. Reflectivity impulse response of a delta spike, slope $m = 10/s$.

In Figure 8, we show the reflection impulse response for the case of a much longer transition zone of 1000m, where the velocities change again from 1000m/s to 2000m/s, for a slope of one. We obtain a much broader reflected pulse, about 1.4 seconds long. It is interesting to observe that 1.4 seconds is about twice the amount of time it takes a signal to propagate across the transition zone at these speeds, which makes physical sense since the impulse, as it is being reflected, has to travel twice across this zone.

The same computation can be performed on the transmission coefficients, resulting in the transmitted waveform shown in Figure 9. Observe there is a much narrower spike in this transmitted case, essentially a delta spike itself.

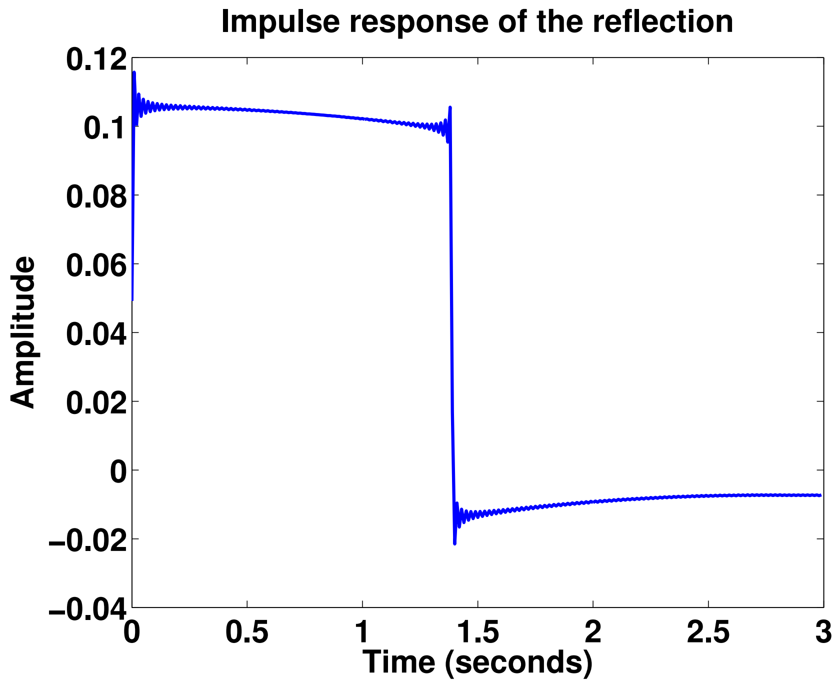

FIG. 8. Reflectivity impulse response of a delta spike, slope $m = 1/s$.

FUTURE WORK

Our next step is to extend this modelling by exact solution to 2D and 3D models, where the transition zone varies linearly in only one dimension. We expect the same qualitative behaviour, and would like to extract information on what frequencies are necessary to observe the linear velocity ramps.

CONCLUSIONS

We have demonstrated an exact analytic solution to the 1D wave equation for a physical medium with a linear velocity ramp sandwiched between two constant velocity regions. From this, we computed reflection and transmission coefficients for the ramp, both numerically and analytically, showing that these coefficients are frequency dependent. Asymptotically, they behave like the coefficients for a jump discontinuity in velocity. The results for varying density, and varying modulus of elasticity are different – reflecting the important physical observation that it is not enough to know only the velocity field. One must know the density and elasticity in the non-constant case. Finally, we numerically computed the transmitted and reflected waveforms that result when a delta spike impinges on the velocity ramp, observed that the reflected waveform is actually broadened into a boxcar. Examples were presented with physically relevant models with realistic physical velocity, density, and modulus of elasticity. We observed that the slow increase in seismic velocities seen in typically sedimentary basins can only been seen in seismic data that faithfully records very low frequencies.

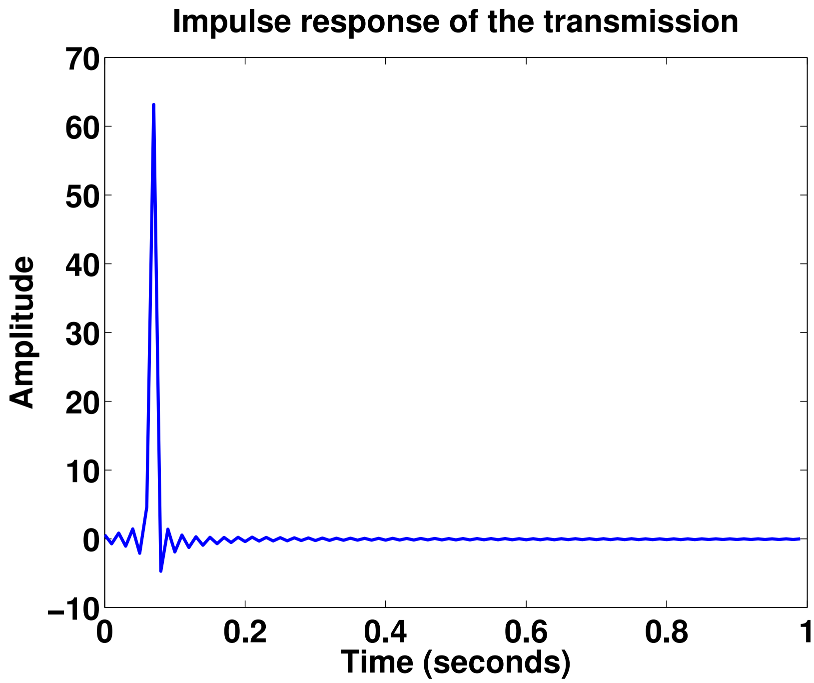

FIG. 9. Transmission impulse response of a delta spike.

ACKNOWLEDGEMENTS

The authors gratefully acknowledge the financial support of NSERC, Mprime, PIMS, and the industrial sponsors of the POTSI and CREWES projects at the University of Calgary.