Poroelastic scattering potentials and inversion sensitivities

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ABSTRACT

Estimating the seismic wavefields response corresponding to the small model parameters' perturbations is a classical problem in inverse scattering problem of exploration geophysics. The Fréchet derivatives or sensitive matrices play a crucial role in perturbation analysis and are considered as sensitivity kernels in least-squares inverse problems. The forward modeling problem in poroelastic media has been studied by many researchers, while the inverse problem for poroelastic media has rarely been investigated. The scattering potentials indicating the perturbations of model parameters can be considered as engines for seismic wave scattering. And they are closely related to the Fréchet derivatives. In this research, we reviewed the Biot's theory for poroelastic wave equations and derived the poroelastic scattering potentials represented by different field variables firstly. And then we derived the coupled poroelastic Fréchet derivatives with respect to 9 poroelastic parameters, namely, the Lamé coefficients of the dry frame λ_{dry} and μ, porosity/fluid term f, density of saturated medium ρ_{sat}, fluid density ρ_f, C, M, $\tilde{\rho}$,and mobility of the fluid m using perturbation method and non-perturbation method. The porosity/fluid term f involved by Russell et al. (2011) for linearized AVO analysis is considered as a poroelastic parameter for sensitivity analysis. The explicit expressions for these Fréchet derivatives with respect to different poroelastic parameters are provided. When wave propagating in poroelastic media, there are two kinds of compressional waves: the fast compressional wave and the slow compressional wave. In this research, we also derived the P-SV Fréchet derivatives in which the fast compressional wave and slow compressional wave are coupled together.

A REVIEW OF BIOT'S THEORY FOR POROELASTIC WAVE EQUATIONS

Biot (Biot, 1955, 1956a,b; Biot and Willis, 1957; Biot, 1962) developed classic theory of the propagation of the stress waves in porous elastic solid containing a compressible viscous fluid. He found that the poroelastic material can be described by four nondimensional parameters and a characteristic frequency. In this section, the concepts of the stress and strain in the aggregate including the fluid pressure and dilatation are reviewed following Biot. For a volume of the porous elastic solid saturated by a viscous fluid system represented by a cube of a unit size. The stress tensor can be separated into two parts: one denotes the stress acting on the solid parts of each face of a cube,

$$\begin{pmatrix} \sigma_{xx} & \varsigma_z & \varsigma_y \\ \varsigma_z & \sigma_{yy} & \varsigma_x \\ \varsigma_y & \varsigma_x & \sigma_{zz} \end{pmatrix} \tag{1}$$

and the other denotes the stress acting on the fluid parts of each cube face,

$$\begin{pmatrix} S & 0 & 0 \\ 0 & S & 0 \\ 0 & 0 & S \end{pmatrix} \tag{2}$$

where S is a scalar and it is proportional to the fluid pressure p, which can be expressed as:

$$S = -\upsilon p, \tag{3}$$

where υ denotes the porosity, which is designated as that which is connected with the bulk motion of the fluid relative to the solid (Biot, 1955). Such that, the stress field of the porous medium can be denoted as:

$$\begin{pmatrix} \sigma_{xx}+S & \varsigma_z & \varsigma_y \\ \varsigma_z & \sigma_{yy}+S & \varsigma_x \\ \varsigma_y & \varsigma_x & \sigma_{zz}+S \end{pmatrix} \tag{4}$$

In this research, the sealed space is considered part of the solid. The strain tensor in the solid can be denoted as:

$$\begin{pmatrix} e_{xx} & \gamma_z & \gamma_y \\ \gamma_z & e_{yy} & \gamma_y \\ \gamma_x & \gamma_y & e_{zz} \end{pmatrix} \tag{5}$$

where

$$\begin{aligned} e_{xx} &= \partial_x u_x, \\ e_{yy} &= \partial_y u_y, \\ e_{zz} &= \partial_z u_z, \\ \gamma_x &= \frac{1}{2}\left(\partial_y u_z + \partial_z u_y\right), \\ \gamma_y &= \frac{1}{2}\left(\partial_z u_x + \partial_x u_z\right), \\ \gamma_z &= \frac{1}{2}\left(\partial_x u_y + \partial_y u_x\right), \\ e &= e_{xx} + e_{yy} + e_{zz}. \end{aligned} \tag{6}$$

where $u_i, i = x, y, z$ is the components of the displacement vector of the solid. These theories are based on the assumptions that the size of the unit elements is very large comparing with the size of the pores and the displacement of the material is uniform and averaged over the element. Similarly, the strain in the fluid can be defined as:

$$\epsilon = \partial_x U_x + \partial_y U_y + \partial_z U_z, \tag{7}$$

where $U_i, i = x, y, z$ is the average fluid displacement vector. And the displacement of the fluid relative to the solid is:

$$\xi = -\left(\partial_x w_x + \partial_y w_y + \partial_z w_z\right), \tag{8}$$

where $w_i = \upsilon\left(U_i - u_i\right), i = x, y, z.$ with the vector notation:

$$\mathbf{u} = \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix}, \mathbf{U} = \begin{pmatrix} U_x \\ U_y \\ U_z \end{pmatrix}, \mathbf{w} = \upsilon\left(\mathbf{U} - \mathbf{u}\right). \tag{9}$$

e, ϵ and ξ are equal to applying divergence operation to **u**, **U** and **w** respectively, which can be written as:

$$e = \nabla\cdot\mathbf{u}, \epsilon = \nabla\cdot\mathbf{U}, \xi = -\nabla\cdot\mathbf{w}. \tag{10}$$

Now we can establish the relationship between the stress and strain of the solid-fluid aggregate. All dissipative forces are disregarded here for convenience, which means that the system is conservative. Further, Biot (1955) assumed that the solid-fluid system is statistically isotropic and took the properties of the symmetry of the material into consideration. Then the complicated stress-strain relations are simplified considerably:

$$\begin{aligned}
&\sigma_{xx} = 2\mu e_{xx} + \lambda_{sat} e - \varphi M \xi,\\
&\sigma_{yy} = 2\mu e_{yy} + \lambda_{sat} e - \varphi M \xi,\\
&\sigma_{zz} = 2\mu e_{zz} + \lambda_{sat} e - \varphi M \xi,\\
&\varsigma_z = \mu \gamma_z,\\
&\varsigma_y = \mu \gamma_y,\\
&\varsigma_x = \mu \gamma_x,\\
&p = -\varphi M e + M\xi,
\end{aligned} \tag{11}$$

where λ and μ are the Lamé coefficients. In abbreviated notation, the above equations can also be written as:

$$\begin{aligned}
&\nu_{ij} = 2\mu e_{ij} + \delta_{ij}\left(\lambda_{sat} e - \varphi M \xi\right),\\
&\delta_{ij} = 1, i = j,\\
&\delta_{ij} = 0, i \neq j.
\end{aligned} \tag{12}$$

In other publications of (Biot, 1956a,b, 1962), the equation denoting fluid pressure can also be written in the form of:

$$S = Qe + R\epsilon, \tag{13}$$

where $R = \upsilon^2 M$ is a measure of the pressure required on the fluid to force a certain volume of the fluid into the aggregate while the total volume remains constant (Biot, 1955) and it is also called Biot elastic coefficient. $Q = \upsilon(\varphi - \upsilon)M$ is the coefficient which couples the volume change of the solid and that of the fluid. If the fluid pressure s is equal to zero, the fluid strain ε can be written as:

$$\varepsilon = -\frac{Qe}{R}, \tag{14}$$

The equations of motion are:

$$\begin{aligned}
&\partial_x \sigma_{xx} + \partial_y \varsigma_z + \partial_z \varsigma_y = -\omega^2\left(\rho_{sat} u_x + \rho_f w_x\right),\\
&\partial_y \sigma_{yy} + \partial_z \varsigma_z + \partial_z \varsigma_x = -\omega^2\left(\rho_{sat} u_y + \rho_f w_y\right),\\
&\partial_z \sigma_{zz} + \partial_y \varsigma_x + \partial_x \varsigma_y = -\omega^2\left(\rho_{sat} u_z + \rho_f w_z\right),\\
&-\omega^2\left(\rho_f u_x + m w_x\right) + i\omega(\eta/\kappa) w_x = \partial_x\left(\varphi M e - M\xi\right),\\
&-\omega^2\left(\rho_f u_y + m w_y\right) + i\omega(\eta/\kappa) w_y = \partial_y\left(\varphi M e - M\xi\right),\\
&-\omega^2\left(\rho_f u_z + m w_z\right) + i\omega(\eta/\kappa) w_z = \partial_z\left(\varphi M e - M\xi\right),
\end{aligned} \tag{15}$$

Then inserting equation (11) into equation (15) and for constant values of the parameters, these equations can be written as:

$$\begin{aligned}
&\mu \nabla^2 \mathbf{u} + \left(\mu + \lambda_{sat}\right) \nabla e - \varphi M \nabla \xi = -\omega^2\left(\rho_{sat}\mathbf{u} + \rho_f \mathbf{w}\right),\\
&\nabla\left(\varphi M e - M\xi\right) = -\omega^2\left(\rho_f \mathbf{u} + m\mathbf{w}\right) + i\omega(\eta/\kappa)\mathbf{w},
\end{aligned} \tag{16}$$

By introducing the operations $\nabla\cdot$ and $\nabla\times$, the shear waves can be uncoupled from the compressional waves and obey independent equations of propagation.

$$\nabla \cdot \mathbf{u} = e, \nabla \cdot \mathbf{w} = -\xi, \nabla \times \mathbf{u} = \mathbf{\Lambda}, \nabla \times \mathbf{w} = \mathbf{\Omega}. \tag{17}$$

Applying the divergence operation on both sides of equation (16) gives:

$$\begin{aligned}\nabla^2\left[(\lambda_{sat}+2\mu)e-\varphi M\xi\right]&=-\omega^2(\rho_{sat}e-\rho_f\xi),\\ \nabla^2(-\varphi Me+M\xi)&=-\omega^2(-\rho_f e+m\xi)+i\omega(\eta/\kappa)\xi,\end{aligned}\tag{18}$$

These two equations correspond to the propagation of two compressional waves. And each of the compressional wave coupled motion in the fluid and the solid. Similarly, by applying the $\nabla\times$ operation on both sides of equation (16), we can get the equations corresponding to the two shear waves:

$$\begin{aligned}\mu\nabla^2\mathbf{\Lambda}&=-\omega^2(\rho_{sat}\mathbf{\Lambda}+\rho_f\mathbf{\Omega}),\\ \omega(\eta/\kappa)\mathbf{\Omega}&=-\omega^2(\rho_f\mathbf{\Lambda}+m\mathbf{\Omega}),\end{aligned}\tag{19}$$

where $m=\rho_f\frac{\tau}{\upsilon}$. And these two shear waves couple the rotation of the solid and that of the fluid. In other publications of Biot (1956a), the compressional waves and shear waves are also written in the form of:

$$\begin{aligned}&\nabla^2\left[(\lambda_{sat}+2\mu)e+Q\epsilon\right]=-\omega^2(\rho_{11}e+\rho_{12}\epsilon),\\ &\nabla^2(Qe+R\epsilon)=-\omega^2(\rho_{12}e+\rho_{22}\epsilon),\\ &-\omega^2(\rho_{11}\mathbf{\Lambda}+\rho_{12}\mathbf{\Gamma})=\mu\nabla^2\mathbf{\Lambda},\\ &-\omega^2(\rho_{12}\mathbf{\Lambda}+\rho_{22}\mathbf{\Gamma})=0,\end{aligned}\tag{20}$$

where $\mathbf{\Gamma}=\nabla\times\mathbf{U}$ and $\epsilon=\nabla\cdot\mathbf{U}$. And ρ_{11}, ρ_{12} and ρ_{22} are the mass coefficients and if there is no relative motion between the solid and fluid, we can get that:

$$\begin{aligned}&\rho_{11}+2\rho_{12}+\rho_{22}=\rho_{sat},\\ &\rho_{11}=\rho_{sat}-2\upsilon\rho_f+m\upsilon^2,\\ &\rho_{12}=\upsilon\rho_f-m\upsilon^2,\\ &\rho_{22}=m\upsilon^2,\end{aligned}\tag{21}$$

The fast compressional wave and slow compressional wave

To discuss the compressional waves, we can introduce a reference velocity α_c firstly. If the relative motion between the fluid and solid were completely prevented in some way, which means that $e=\epsilon$ in equation (16). Then we can get the reference velocity of a compressional wave:

$$V_c^2=\frac{H}{\rho_{sat}},\tag{22}$$

where $H=P+R+2Q$ and $P=\lambda_{sat}+2\mu$. The solutions of equation (20) can be written in the form (Biot, 1956a):

$$\begin{aligned}e&=C_1e^{i(lx+\alpha t)},\\ \epsilon&=C_2e^{i(lx+\alpha t)}.\end{aligned}\tag{23}$$

Then the velocities can be determined by inserting equation (23) into equation (20)(Biot, 1956a):

$$V_i=\frac{V_c^2}{z_i},\tag{24}$$

where

$$z_i = \frac{\frac{\rho_{11}}{\rho_{sat}} C_1^i + 2\frac{\rho_{12}}{\rho_{sat}} C_1^i C_2^i + \frac{\rho_{22}}{\rho_{sat}} C_2^i}{\frac{P}{H} C_1^i + 2\frac{Q}{H} C_1^i C_2^i + \frac{R}{H} C_2^i},$$

where $i = 1, 2$ indicates the fast P-wave and slow P-wave respectively. We can also find the solutions for fast P-wave and slow P-wave in the papers by Morency and Tromp (2008), Morency et al. (2009) and Yeh et al. (2004).

The shear wave

Considering equation (17), to get the shear waves, we can eliminate $\mathbf{\Gamma}$ in the equations, which gives:

$$\mu \nabla^2 \mathbf{\Lambda} = -\omega^2 \rho_{11} \left(1 - \frac{\rho_{12}^2}{\rho_{11}\rho_{12}} \right) \mathbf{\Lambda}, \tag{25}$$

We can see that there is only one shear wave exists and the velocity of this S-wave can be expressed as:

$$\beta = \sqrt{\frac{\mu_{sat}}{\rho_{11} \left(1 - \frac{\rho_{12}^2}{\rho_{11}\rho_{12}} \right)}}, \tag{26}$$

POROELASTIC SCATTERING POTENTIALS

The poroelastic wave equations were obtained through Biot's pioneering work (Biot, 1956a,b; Biot and Willis, 1957; Biot, 1962), just as what we have discussed above. The solutions of the two wave equations have been studied by many authors. However, the Green's functions can be different because of different combinations of field variables. One set of field variables used to express the poroelastic wave equations are average solid displacements **u** and relative fluid-solid displacements **w** and another method is using average solid displacements **u** and fluid displacements **U**. In this section, we will extend the 3D isotropic elastic methods of Stolt and Weglein (2012) to treat the problem of poroelastic scattering , invoking each of the two representations in turn.

Solid and Relative fluid-solid Displacement Representation

Recall the poroelastic wave equations represented using solid displacements **u** and relative fluid-solid displacements **w**:

$$\begin{aligned} &\nabla \cdot (\lambda_{sat} \nabla \cdot \mathbf{u} + C \nabla \cdot \mathbf{w}) \mathbf{I} + 2\mu \nabla^2 \mathbf{u} + \mathbf{F} = -\omega^2 (\rho_{sat} \mathbf{u} + \rho_f \mathbf{w}), \\ &\nabla \cdot (C \nabla \cdot \mathbf{u} + M \nabla \cdot \mathbf{w}) \mathbf{I} + \mathbf{f} = -\omega^2 (\rho_f \mathbf{u} + \tilde{\rho} \mathbf{w} + m \mathbf{w}), \end{aligned} \tag{27}$$

where **I** is the identity tensor and $\tilde{\rho} = \omega \frac{i\eta}{k}$. And the poroelastic governing wave equations can be expressed in matrix form (Karpfinger et al., 2009):

$$\boldsymbol{\mathcal{L}}_P(\mathbf{r}, \omega) \cdot \begin{pmatrix} \mathbf{u}(\mathbf{r}, \omega) \\ \mathbf{w}(\mathbf{r}, \omega) \end{pmatrix} = - \begin{pmatrix} \mathbf{F} \\ \mathbf{f} \end{pmatrix}, \tag{28}$$

where $\mathbf{r} = (x, y, z)$ indicates the subsurface position, $\mathbf{u} = \begin{pmatrix} u_x & u_y & u_z \end{pmatrix}^T$, $\mathbf{w} = \begin{pmatrix} w_x & w_y & w_z \end{pmatrix}^T$, $\mathbf{F} = F(\omega)\delta(\mathbf{r} - \mathbf{r}')$ and $\mathbf{f} = f(\omega)\delta(\mathbf{r} - \mathbf{r}')$ are the sources applied to solid and fluid phases

Table1. Nomenclature, listed as introduced in the text

Symbol	Description
$\mathbf{r} = (x, y, z)$	Spatial coordinates (m)
ω	Temporal frequency
$\lambda_{sat}, \lambda_{dry}$	Lam$\acute{e}$ coefficients of the saturated material and skeletal frame
μ_{sat}, μ_{dry}	Shear modulus of the saturated material and skeletal frame
$K_{sat}, K_{dry}, K_s, K_f$	Bulk modulus of the saturated material, dry frame, solid material, and fluid
$\rho_{sat}, \rho_s, \rho_f$	Densities of the saturated material, solid material, and fluid
υ	Porosity of the saturated material
κ	Permeability of material
η	Viscosity of the fluid
τ	Turtosity of the matrix
$\varphi = 1 - \frac{K_{dry}}{K_s}$	Biot-Willis coefficient (Biot and Willis, 1957)
$M^{-1} = \frac{\varphi - \upsilon}{K_s} + \frac{\upsilon}{K_f}$	Pore space modulus
$f = \varphi^2 M$	Fluid/porosity term
$\alpha_{sat}, \beta_{sat}$	Compressional and shear velocities of the saturated material
p	Fluid pressure
$\mathbf{u} = \begin{pmatrix} u_x & u_y & u_z \end{pmatrix}^T$	The average solid displacement vector
$\mathbf{U} = \begin{pmatrix} U_x & U_y & U_z \end{pmatrix}^T$	The average fluid displacement vector
$\mathbf{w} = \upsilon(\mathbf{U} - \mathbf{u})$	Relative fluid to solid displacements
$m = \frac{\rho_f \tau}{\upsilon}$	Mobility of the fluid
$Q = \upsilon(\varphi - \upsilon)M$	The coefficient which couples the volume change of the solid and that of the fluid
$R = \upsilon^2 M$	Biot elastic coefficient
$C = \varphi M$	
$b = \frac{\eta}{\kappa}$	

respectively. $F(\omega)$ and $f(\omega)$ are the source signatures, and $\delta(\mathbf{r}-\mathbf{r}')$ is the Dirac delta function. $\boldsymbol{\mathcal{L}}_P$ is a 6×6 matrix:

$$\boldsymbol{\mathcal{L}}_P(\mathbf{r},\omega)=\begin{pmatrix}\boldsymbol{\mathcal{L}}^{s1}(\mathbf{r},\omega) & \boldsymbol{\mathcal{L}}^{f1}(\mathbf{r},\omega)\\ \boldsymbol{\mathcal{L}}^{s2}(\mathbf{r},\omega) & \boldsymbol{\mathcal{L}}^{f2}(\mathbf{r},\omega)\end{pmatrix} \tag{29}$$

where $\boldsymbol{\mathcal{L}}^{s1}$, $\boldsymbol{\mathcal{L}}^{s2}$, $\boldsymbol{\mathcal{L}}^{f1}$ and $\boldsymbol{\mathcal{L}}^{f2}$ are all 3×3 matrices and they can be written as:

$$\begin{aligned}\boldsymbol{\mathcal{L}}^{s1}&=\begin{pmatrix}L^{s1}_{xx} & L^{s1}_{xy} & L^{s1}_{xz}\\ L^{s1}_{yx} & L^{s1}_{yy} & L^{s1}_{yz}\\ L^{s1}_{zx} & L^{s1}_{zy} & L^{s1}_{zz}\end{pmatrix},\boldsymbol{\mathcal{L}}^{s2}=\begin{pmatrix}L^{s2}_{xx} & L^{s2}_{xy} & L^{s2}_{xz}\\ L^{s2}_{yx} & L^{s2}_{yy} & L^{s2}_{yz}\\ L^{s2}_{zx} & L^{s2}_{zy} & L^{s2}_{zz}\end{pmatrix},\\ \boldsymbol{\mathcal{L}}^{f1}&=\begin{pmatrix}L^{f1}_{xx} & L^{f1}_{xy} & L^{f1}_{xz}\\ L^{f1}_{yx} & L^{f1}_{yy} & L^{f1}_{yz}\\ L^{f1}_{zx} & L^{f1}_{zy} & L^{f1}_{zz}\end{pmatrix},\boldsymbol{\mathcal{L}}^{f2}=\begin{pmatrix}L^{f2}_{xx} & L^{f2}_{xy} & L^{f2}_{xz}\\ L^{f2}_{yx} & L^{f2}_{yy} & L^{f2}_{yz}\\ L^{f2}_{zx} & L^{f2}_{zy} & L^{f2}_{zz}\end{pmatrix}.\end{aligned} \tag{30}$$

And we can notice that $\boldsymbol{\mathcal{L}}^{s2}=\boldsymbol{\mathcal{L}}^{f1}$ and each element in these matrices can be written as:

$$\begin{aligned}&L^{s1}_{ii}=\partial_i\lambda_{dry}\partial_i+\partial_i f\partial_i+2\partial_i\mu\partial_i+\sum_{j\neq i}\partial_j\mu\partial_j+\rho_{sat}\omega^2, i,j=x,y,z;\\ &L^{s1}_{ij}=\partial_i\lambda_{dry}\partial_j+\partial_i f\partial_j+\partial_j\mu\partial_i, i\neq j.\\ &L^{sf}_{ii}=L^{f1}_{ii}=L^{s2}_{ii}=\partial_i C\partial_i+\rho_f\omega^2, i,j=x,y,z;\\ &L^{sf}_{ii}=L^{s2}_{ij}=\partial_i C\partial_j, i\neq j.\\ &L^{f2}_{ii}=-\partial_i M\partial_i+\tilde{\rho}\omega-m\omega^2, i,j=x,y,z;\\ &L^{f2}_{ij}=-\partial_i M\partial_j, i\neq j.\end{aligned} \tag{31}$$

The poroelastic scattering potentials $\boldsymbol{\mathcal{V}}_P$ are the difference between the perturbed wave operator $\boldsymbol{\mathcal{L}}_P$ and unperturbed wave operator $\boldsymbol{\mathcal{L}}^0_P$:

$$\boldsymbol{\mathcal{V}}_{\boldsymbol{P}}=\boldsymbol{\mathcal{L}}_P-\boldsymbol{\mathcal{L}}^0_P, \tag{32}$$

The poroelastic scattering potentials are also a 6×6 matrix, which can be written as:

$$\boldsymbol{\mathcal{V}}_{\boldsymbol{P}}=\begin{pmatrix}\boldsymbol{\mathcal{V}}^{s1} & \boldsymbol{\mathcal{V}}^{sf}\\ \boldsymbol{\mathcal{V}}^{sf} & \boldsymbol{\mathcal{V}}^{f2}\end{pmatrix} \tag{33}$$

where $\boldsymbol{\mathcal{V}}^{s1}$, $\boldsymbol{\mathcal{V}}^{sf}$ and $\boldsymbol{\mathcal{V}}^{f2}$ are all 3×3 matrices and the elements in these matrices can be written as:

$$V^{s1}_{ii}=\lambda^0_{dry}\partial_i a_{\lambda_{dry}}\partial_i+f^0\partial_i a_f\partial_i+\rho^0_{sat}\left(a_\rho\omega^2+(\beta^0_{sat})^2\partial_i a_\mu\partial_i+(\beta^0_{sat})^2\sum_{j\neq i}\partial_j a_\mu\partial_j\right), i,j=x,y,z,$$

$$\begin{aligned}&V^{s1}_{ij}=(\lambda^0_{dry})^2\partial_i a_{\lambda_{dry}}\partial_j+f^0\partial_i a_f\partial_j+\rho^0_{sat}(\beta^0_{sat})^2\partial_j a_\mu\partial_i, i\neq j.\\ &V^{sf}_{ii}=C_0\partial_i a_c\partial_i+\rho^0_f a_{\rho_f}\omega^2, i,j=x,y,z;\\ &V^{sf}_{ij}=C_0\partial_i a_c\partial_j, i\neq j.\\ &V^{f2}_{ii}=-m_0a_m\omega^2-M_0\partial_i a_M\partial_i+\tilde{\rho}^0a_{\tilde{\rho}}\omega^2, i,j=x,y,z;\\ &V^{f2}_{ij}=-M_0\partial_i a_M\partial_j, i\neq j.\end{aligned} \tag{34}$$

where

$$\begin{aligned}
a_\rho &= \frac{\rho_{sat}-\rho_{sat}^0}{\rho_{sat}^0} \simeq \frac{\triangle\rho_{sat}}{\rho_{sat}};\\
a_{\lambda_{dry}} &= \frac{\lambda_{dry}-\lambda_{dry}^0}{\lambda_{dry}^0} \simeq \frac{\triangle\lambda_{dry}}{\lambda_{dry}};\\
a_f &= \frac{f-f^0}{f^0} \simeq \frac{\triangle f}{f};\\
a_\mu &= \frac{\mu-\mu^0}{\mu^0} \simeq \frac{\triangle\mu}{\mu};\\
a_\rho^f &= \frac{\rho_f-\rho_f^0}{\rho_f^0} \simeq \frac{\triangle\rho_f}{\rho_f};\\
a_C &= \frac{C-C^0}{C^0} \simeq \frac{\triangle C}{C};\\
a_m &= \frac{m-m^0}{m^0} \simeq \frac{\triangle m}{m};\\
a_M &= \frac{M-M^0}{M^0} \simeq \frac{\triangle M}{M};\\
a_{\tilde\rho} &= \frac{\tilde\rho-\tilde\rho^0}{\tilde\rho^0} \simeq \frac{\triangle\tilde\rho^0}{\tilde\rho^0}.
\end{aligned} \tag{35}$$

Solid and fluid displacement representation

In this part, we will express the poroelastic scattering potentials using the solid displacements $\mathbf{u}$ and the relative fluid displacements $\mathbf{U}$. Recall the poroelastic wave equations:

$$\nabla\cdot\left(\lambda_{sat}\nabla\cdot\mathbf{u}+Q\nabla\cdot\mathbf{U}\right)\mathbf{I}+2\mu\nabla^2\mathbf{u}+\mathbf{F} = -\omega^2\left(\rho_{11}\mathbf{u}+\rho_{22}\mathbf{U}\right), \tag{36}$$

$$\nabla\cdot\left(Q\nabla\cdot\mathbf{u}+R\nabla\cdot\mathbf{U}\right)\mathbf{I}+\mathbf{f} = -\omega^2\left(\rho_{11}\mathbf{u}+\rho_{12}\mathbf{U}\right).$$

Similarly, it can written in a matrix form:

$$\begin{pmatrix} \boldsymbol{\mathcal{L}}^{s1}(\mathbf{r},\omega) & \boldsymbol{\mathcal{L}}^{sf}(\mathbf{r},\omega) \\ \boldsymbol{\mathcal{L}}^{sf}(\mathbf{r},\omega) & \boldsymbol{\mathcal{L}}^{f2}(\mathbf{r},\omega) \end{pmatrix} \cdot \begin{pmatrix} \mathbf{u}(\mathbf{r},\omega) \\ \mathbf{w}(\mathbf{r},\omega) \end{pmatrix} = -\begin{pmatrix} \mathbf{F} \\ \mathbf{f} \end{pmatrix}, \tag{37}$$

And the elements in differential matrix can be written as:

$$\begin{aligned}
L_{ii}^{s1} &= \partial_i\lambda_{dry}\partial_i + 2\partial_i\mu\partial_i + \partial_i f\partial_i + \sum_{i\neq j}\partial_j\mu\partial_j + \rho_{11}\omega^2, i,j = x,y,z;\\
L_{ij}^{s1} &= \partial_i\lambda_{dry}\partial_j + \partial_i f\partial_j + \partial_j\mu\partial_i, i\neq j.\\
L_{ii}^{sf} &= \partial_i Q\partial_i + \rho_{12}\omega^2, i,j = x,y,z;\\
L_{ij}^{sf} &= \partial_i Q\partial_j, i\neq j.\\
L_{ii}^{f2} &= \partial_i R\partial_i + \rho_{22}\omega^2, i,j = x,y,z;\\
L_{ij}^{f2} &= \partial_i R\partial_j, i\neq j.
\end{aligned} \tag{38}$$

And the corresponding scattering potentials are:

$$
\begin{aligned}
V_{ii}^{s1} &= \rho_{11}^0 a_{\rho_{11}}\omega^2 + \lambda_{dry}^0 \partial_i a_{\lambda_{dry}}\partial_i + 2\mu^0\partial_i a_\mu \partial_i + \mu_0 \sum_{i\neq j}\partial_j a_\mu \partial_j, i,j = x,y,z;\\
V_{ij}^{s1} &= \lambda_{dry}^0 \partial_i a_{\lambda_{dry}}\partial_j + f^0\partial_i a_f \partial_j + \mu_0 \partial_i a_\mu \partial_j, i \neq j.\\
V_{ii}^{sf} &= \rho_{12}^0 a_{\rho_{12}}\omega^2 + Q_0 \partial_i a_Q \partial_i, i,j = x,y,z;\\
V_{ij}^{sf} &= Q_0 \partial_i a_Q \partial_j, i \neq j.\\
V_{ii}^{f2} &= \rho_{22}^0 a_{\rho_{22}}\omega^2 + Q_0 \partial_i a_R \partial_i, i,j = x,y,z;\\
V_{ij}^{f2} &= R_0 \partial_i a_R \partial_j, i \neq j.
\end{aligned}
\tag{39}
$$

where

$$
\begin{aligned}
a_{\rho_{11}} &= \frac{\rho_{11}-\rho_{11}^0}{\rho_{11}^0} \simeq \frac{\triangle\rho_{11}}{\rho_{11}};\\
a_{\rho_{12}} &= \frac{\rho_{12}-\rho_{12}^0}{\rho_{12}^0} \simeq \frac{\triangle\rho_{12}}{\rho_{12}};\\
a_{\rho_{22}} &= \frac{\rho_{22}-\rho_{22}^0}{\rho_{22}^0} \simeq \frac{\triangle\rho_{22}}{\rho_{22}};\\
a_{\lambda_{dry}} &= \frac{\lambda_{dry}-\lambda_{dry}^0}{\lambda_{dry}^0} \simeq \frac{\triangle\lambda_{dry}}{\lambda_{dry}};\\
a_f &= \frac{f-f^0}{f^0} \simeq \frac{\triangle f}{f};\\
a_\mu &= \frac{\mu-\mu^0}{\mu^0} \simeq \frac{\triangle\mu}{\mu};\\
a_Q &= \frac{Q-Q^0}{Q^0} \simeq \frac{\triangle Q}{Q};\\
a_R &= \frac{R-R^0}{R^0} \simeq \frac{\triangle R}{R}.
\end{aligned}
\tag{40}
$$

INVERSION SENSITIVITIES

Recent developments in least-squares inverse problem has stimulated new interests in the classic problem in exploration geophysics, which is the estimation of the sensitivity of the seismic wavefield response corresponding to the small perturbations in the model properties. The sensitivity operator, which is often referred as the Fr$\acute{e}$chet derivative, plays a crucial role in the least-squares inverse problems (Tarantola and Valette, 1982; Dietrich and Kormendi, 1990), such as Full Waveform Inversion (FWI) and Least-squares Migration (LSM).

Scattering Potential and Fr$\acute{e}$chet Derivative

The Fr$\acute{e}$chet derivatives are always introduced to express the inversion sensitivities in forward and inverse scattering problems which begin with a wave equation operator $\mathcal{L}$ and a Green's function $\mathbf{G}$:

$$\boldsymbol{\mathcal{L}}(\mathbf{r}_g, \omega)\mathbf{G}(\mathbf{r}_g, \mathbf{r}_s, \omega) = \delta(\mathbf{r}_g - \mathbf{r}_s), \tag{41}$$

where $\boldsymbol{\mathcal{L}}$ is also called the differential operator, $\mathbf{r}_g = (x_g, y_g, z_g)$ and $\mathbf{r}_s = (x_s, y_s, z_s)$ indicate the receivers' locations and sources' locations respectively. We can define the unperturbed and perturbed wave equations as follows:

$$\boldsymbol{\mathcal{L}}_0(\mathbf{r}_g, \omega)\mathbf{G}_0(\mathbf{r}_g, \mathbf{r}_s, \omega) = \delta(\mathbf{r}_g - \mathbf{r}_s), \tag{42}$$

$$\boldsymbol{\mathcal{L}}(\mathbf{r}_g, \omega)\mathbf{G}(\mathbf{r}_g, \mathbf{r}_s, \omega) = \delta(\mathbf{r}_g - \mathbf{r}_s), \tag{43}$$

where $\boldsymbol{\mathcal{L}} = \boldsymbol{\mathcal{L}}_0 + \delta\boldsymbol{\mathcal{L}}$, and $\delta\boldsymbol{\mathcal{L}}$ is the model perturbation which is identical to the scattering potential $\boldsymbol{\mathcal{V}}$.

Fréchet Derivative: Perturbation Derivation

Substituting perturbed wave modeling operator $\boldsymbol{\mathcal{L}} = \boldsymbol{\mathcal{L}}_0 + \boldsymbol{\mathcal{V}}$ into equation (51) gives:

$$(\boldsymbol{\mathcal{L}}_0 + \boldsymbol{\mathcal{V}})\,\mathbf{G} = \delta(\mathbf{r}_g - \mathbf{r}_s), \tag{44}$$

Isolating the Green's function $\mathbf{G}$ in the perturbed medium on the left hand side of the equation forms the classical Lippmann-Schwinger equation (Newton, 1966; Taylor, 1972; Stolt and Weglein, 2012):

$$\mathbf{G} = \mathbf{G}_0 + \mathbf{G}_0\boldsymbol{\mathcal{V}}\mathbf{G} \tag{45}$$

The wavefield response $\delta\mathbf{G} = \mathbf{G} - \mathbf{G}_0$ corresponding to the model perturbation can be formulated in a series in the quantity $\mathbf{G}_0\boldsymbol{\mathcal{V}}$ (Innanen, 2008; Stolt and Weglein, 2012):

$$\delta\mathbf{G} = \sum_{n=1}^{\infty} \mathbf{G}_0\,(\boldsymbol{\mathcal{V}}\mathbf{G}_0)^n = \mathbf{G}_0\,(\boldsymbol{\mathcal{V}}\mathbf{G}_0)^1 + \mathbf{G}_0\,(\boldsymbol{\mathcal{V}}\mathbf{G}_0)^2 + \mathbf{G}_0\,(\boldsymbol{\mathcal{V}}\mathbf{G}_0)^3 + \ldots, \tag{46}$$

When considering a small scattering potential $\boldsymbol{\mathcal{V}}$ or the norm of the operator $\mathbf{G}_0\boldsymbol{\mathcal{V}}$ is smaller than 1, the high order terms in the above equation can be ignored:

$$\delta\mathbf{G} \simeq \mathbf{G}_0\boldsymbol{\mathcal{V}}\mathbf{G}_0, \tag{47}$$

So, the Fréchet derivatives can be expressed as:

$$\frac{\delta\mathbf{G}}{\delta\mathbf{s}} = \mathbf{G}_0\frac{\boldsymbol{\mathcal{V}}}{\delta\mathbf{s}}\mathbf{G}_0, \tag{48}$$

Fréchet Derivative: Non-Perturbation Derivation

Reexamine the perturbed and unperturbed wave equations (50) and (51), all of the left hand side terms are functions of model parameters, while the source term on the right hand side is not. Taking partial derivative on both sides of the wave equation with respect to the model parameters $\mathbf{s}$ gives:

$$\boldsymbol{\mathcal{L}}_0\frac{\partial\mathbf{G}}{\partial\mathbf{s}} = -\frac{\partial\boldsymbol{\mathcal{V}}}{\partial\mathbf{s}}\mathbf{G}, \tag{49}$$

Substituting $\mathbf{G}_0$ for $\mathbf{G}$ in the above equation forms the single scattering or Born approximation under the assumption of small model perturbation:

$$\boldsymbol{\mathcal{L}}_0 \frac{\partial \mathbf{G}}{\partial \mathbf{s}} \simeq -\frac{\partial \boldsymbol{\mathcal{V}}}{\partial \mathbf{s}} \mathbf{G}_0, \tag{50}$$

The right hand side of the above equation is always referred to as "scattered sources" or "secondary Born sources". It underlines the fact that the scattered wavefields $\delta\mathbf{G}$ due to the perturbations in the model parameters such as density $\delta\rho$, Lam$\acute{e}$ coefficients $\delta\lambda$ and δK, can be interpreted as the wavefield generated by a set of secondary body forces, which propagate in the current, unperturbed medium $\boldsymbol{\mathcal{L}}_0$ (Dietrich and Kormendi, 1990). The inversion sensitivities or perturbation analysis is to compute the Fr$\acute{e}$chet derivatives for the slight perturbations of various model parameters. Because the wave modeling operator $\boldsymbol{\mathcal{L}}_0$ can be expressed using $-\mathbf{G}_0^{-1}$, the Fr$\acute{e}$chret derivative can be expressed as:

$$\frac{\partial \mathbf{G}}{\partial \mathbf{s}} \simeq \mathbf{G}_0 \frac{\partial \boldsymbol{\mathcal{V}}}{\partial \mathbf{s}} \mathbf{G}_0, \tag{51}$$

Poroelastic Fr$\acute{e}$chet Derivatives

I: Perturbation Derivation

In this part, we derived the coupled poroelastic Fr$\acute{e}$chet derivatives using the field variables solid displacement $\mathbf{u}$ and relative fluid-solid displacement $\mathbf{w}$. The poroelastic wave equations in equation (27) can be formulated in matrix form:

$$\begin{pmatrix} L_{ij}^{s1}(\mathbf{r},\omega) & L_{ij}^{sf}(\mathbf{r},\omega) \\ L_{ij}^{sf}(\mathbf{r},\omega) & L_{ij}^{f2}(\mathbf{r},\omega) \end{pmatrix} \cdot \begin{pmatrix} u_j(\mathbf{r},\omega) \\ w_j(\mathbf{r},\omega) \end{pmatrix} = \begin{pmatrix} F_j\delta(\mathbf{r}-\mathbf{r}') \\ f_j\delta(\mathbf{r}-\mathbf{r}') \end{pmatrix} \tag{52}$$

where $i = x, y, z$. And the solutions of the displacements can be obtained in the integral representation (Pride and Haartsen, 1996; Muller and Gurevich, 2005):

$$\begin{pmatrix} u_i(\mathbf{r},\omega) \\ w_i(\mathbf{r},\omega) \end{pmatrix} = \int_\Omega d^3\mathbf{r}' \begin{pmatrix} G_{ij}^{s1}(\mathbf{r}',\omega) & G_{ij}^{sf}(\mathbf{r}',\omega) \\ G_{ij}^{sf}(\mathbf{r}',\omega) & G_{ij}^{f2}(\mathbf{r}',\omega) \end{pmatrix} \cdot \begin{pmatrix} F_j\delta(\mathbf{r}-\mathbf{r}') \\ f_j\delta(\mathbf{r}-\mathbf{r}') \end{pmatrix} \tag{53}$$

If we define the unperturbed displacements as $\begin{pmatrix} \mathbf{u}^0 \\ \mathbf{w}^0 \end{pmatrix}$ and the perturbed displacements as $\begin{pmatrix} \mathbf{u} \\ \mathbf{w} \end{pmatrix}$, the basic poroelastic scattering equation can be written as:

$$\begin{pmatrix} u_i \\ w_i \end{pmatrix} = \begin{pmatrix} u_i^0 \\ w_i^0 \end{pmatrix} + \int_{\boldsymbol{\Omega}} dV \begin{pmatrix} G_{ij}^{s1} & G_{ij}^{sf} \\ G_{ij}^{sf} & G_{ij}^{f2} \end{pmatrix} \cdot \begin{pmatrix} V_{jk}^{s1} & V_{jk}^{sf} \\ V_{jk}^{sf} & V_{jk}^{f2} \end{pmatrix} \cdot \begin{pmatrix} u_k \\ w_k \end{pmatrix} \tag{54}$$

where $\boldsymbol{\Omega}$ indicates the three dimentional volume and $j, k = x, y, z$. The unperturbed and perturbed medium can be denoted using the Greens's functions. And the Green's functions for the inhomogeneous medium can be written as:

$$\begin{pmatrix} G_{il}^{s1} & G_{il}^{sf} \\ G_{il}^{sf} & G_{il}^{f2} \end{pmatrix} = \begin{pmatrix} {}^0G_{il}^{s1} & {}^0G_{il}^{sf} \\ {}^0G_{il}^{sf} & {}^0G_{il}^{f2} \end{pmatrix} + \int_\Omega dV \begin{pmatrix} {}^0G_{ij}^{s1} & {}^0G_{ij}^{sf} \\ {}^0G_{ij}^{sf} & {}^0G_{ij}^{f2} \end{pmatrix} \cdot \begin{pmatrix} V_{jk}^{s1} & V_{jk}^{sf} \\ V_{jk}^{sf} & V_{jk}^{f2} \end{pmatrix} \cdot \begin{pmatrix} G_{kl}^{s1} & G_{kl}^{sf} \\ G_{kl}^{sf} & G_{kl}^{f2} \end{pmatrix} \tag{55}$$

And the above equation can be formulated in a shothand notation:

$$\mathbf{G} = \mathbf{G}_0 + \int_\Omega dV \mathbf{G}_0 \boldsymbol{\mathcal{V}} \mathbf{G}, \tag{56}$$

And it can be expanded as the scattering series:

$$\mathbf{G} = \mathbf{G}_0 + \int \mathbf{G}_0 \boldsymbol{\mathcal{V}} \mathbf{G}_0 + \underbrace{\int\int \mathbf{G}_0 \boldsymbol{\mathcal{V}} \mathbf{G}_0 \boldsymbol{\mathcal{V}} \mathbf{G}_0 + \ldots,}_{NONLINEAR} \tag{57}$$

According to the Born approximation which assumes the week inhomogeneity, we can ignore the nonlinear terms in the scattering series which is equivalent to replacing the Green's function $\mathbf{G}$ with the Green's function $\mathbf{G}_0$ in equation (71):

$$\delta \mathbf{G} \simeq \int_\Omega dV \mathbf{G}_0 \boldsymbol{\mathcal{V}} \mathbf{G}_0, \tag{58}$$

We can notice that the scattered wavefields $\delta\mathbf{G}$ can be presented by the volume integrals with kernels involving the Green's tensors $\mathbf{G}_0$ and the scattered source. The scattered source is composed of the scattering potentials and the Green's funtcion in the unperturbed medium. And the scattering equation becomes:

$$\begin{pmatrix} \delta G_{il}^{s1} & \delta G_{il}^{sf} \\ \delta G_{il}^{sf} & \delta G_{il}^{f2} \end{pmatrix} \simeq \int_\Omega dV \begin{pmatrix} {}^0G_{ij}^{s1} & {}^0G_{ij}^{sf} \\ {}^0G_{ij}^{sf} & {}^0G_{ij}^{f2} \end{pmatrix} \cdot \begin{pmatrix} V_{jk}^{s1} & V_{jk}^{sf} \\ V_{jk}^{sf} & V_{jk}^{f2} \end{pmatrix} \cdot \begin{pmatrix} {}^0G_{kl}^{s1} & {}^0G_{kl}^{sf} \\ {}^0G_{kl}^{sf} & {}^0G_{kl}^{f2} \end{pmatrix}, \tag{59}$$

where $i, j, k, l = x, y, z$. And dividing model perturbations on both sides of the scattering equation gives:

$$\begin{pmatrix} \dfrac{\delta G_{il}^{s1}}{\delta s} & \dfrac{\delta G_{il}^{sf}}{\delta s} \\ \dfrac{\delta G_{il}^{sf}}{\delta s} & \dfrac{\delta G_{il}^{f2}}{\delta s} \end{pmatrix} \simeq \int_\Omega dV \begin{pmatrix} {}^0G_{ij}^{s1} & {}^0G_{ij}^{sf} \\ {}^0G_{ij}^{sf} & {}^0G_{ij}^{f2} \end{pmatrix} \cdot \begin{pmatrix} \dfrac{V_{jk}^{s1}}{\delta s} & \dfrac{V_{jk}^{sf}}{\delta s} \\ \dfrac{V_{jk}^{sf}}{\delta s} & \dfrac{V_{jk}^{f2}}{\delta s} \end{pmatrix} \cdot \begin{pmatrix} {}^0G_{kl}^{s1} & {}^0G_{kl}^{sf} \\ {}^0G_{kl}^{sf} & {}^0G_{kl}^{f2} \end{pmatrix}, \tag{60}$$

II: Non-Perturbation Derivation

The perturbed poroelastic wave equations can be written as:

$$\begin{pmatrix} {}^0L_{ij}^{s1} & {}^0L_{ij}^{sf} \\ {}^0L_{ij}^{sf} & {}^0L_{ij}^{f2} \end{pmatrix} \cdot \begin{pmatrix} u_j \\ w_j \end{pmatrix} = \begin{pmatrix} F_j \delta(\mathbf{r} - \mathbf{r}') \\ f_j \delta(\mathbf{r} - \mathbf{r}') \end{pmatrix} \tag{61}$$

Takaing partial derivative with respect to model parameters on both sides of the equation gives:

$$\begin{pmatrix} \dfrac{\partial^0 L_{jk}^{s1}}{\partial s} & \dfrac{\partial^0 L_{jk}^{sf}}{\partial s} \\ \dfrac{\partial^0 L_{jk}^{sf}}{\partial s} & \dfrac{\partial^0 L_{jk}^{f2}}{\partial s} \end{pmatrix} \cdot \begin{pmatrix} u_k \\ w_k \end{pmatrix} + \begin{pmatrix} {}^0L_{ij}^{s1} & {}^0L_{ij}^{sf} \\ {}^0L_{ij}^{sf} & {}^0L_{ij}^{f2} \end{pmatrix} \cdot \begin{pmatrix} \dfrac{\partial u_j}{\partial s} \\ \dfrac{\partial w_j}{\partial s} \end{pmatrix} = 0, \tag{62}$$

And replacing the displacements using the Green's tensors, as shown by equation (61):

$$-\begin{pmatrix} {}^0L^{s1}_{ij} & {}^0L^{sf}_{ij} \\ {}^0L^{sf}_{ij} & {}^0L^{f2}_{ij} \end{pmatrix} \cdot \begin{pmatrix} \dfrac{\partial G^{s1}_{jl}}{\partial s} & \dfrac{\partial G^{sf}_{jl}}{\partial s} \\ \dfrac{\partial G^{sf}_{jl}}{\partial s} & \dfrac{\partial G^{f2}_{jl}}{\partial s} \end{pmatrix} = \begin{pmatrix} \dfrac{V^{s1}_{jk}}{\partial s} & \dfrac{V^{sf}_{jk}}{\partial s} \\ \dfrac{V^{sf}_{jk}}{\partial s} & \dfrac{V^{f2}_{jk}}{\partial s} \end{pmatrix} \cdot \begin{pmatrix} G^{s1}_{kl} & G^{sf}_{kl} \\ G^{sf}_{kl} & G^{f2}_{kl} \end{pmatrix}, \tag{63}$$

Isoloating the Fr$\acute{e}$chet derivatives ob the left hand side of the equation:

$$\begin{pmatrix} \dfrac{\partial G^{s1}_{il}}{\partial s} & \dfrac{\partial G^{sf}_{il}}{\partial s} \\ \dfrac{\partial G^{sf}_{il}}{\partial s} & \dfrac{\partial G^{f2}_{il}}{\partial s} \end{pmatrix} = -\begin{pmatrix} {}^0L^{s1}_{ij} & {}^0L^{sf}_{ij} \\ {}^0L^{sf}_{ij} & {}^0L^{f2}_{ij} \end{pmatrix}^{-1} \cdot \begin{pmatrix} \dfrac{V^{s1}_{jk}}{\partial s} & \dfrac{V^{sf}_{jk}}{\partial s} \\ \dfrac{V^{sf}_{jk}}{\partial s} & \dfrac{V^{f2}_{jk}}{\partial s} \end{pmatrix} \cdot \begin{pmatrix} G^{s1}_{kl} & G^{sf}_{kl} \\ G^{sf}_{kl} & G^{f2}_{kl} \end{pmatrix}, \tag{64}$$

The inverse of the wave modeling operator can be replaced using the Green's tensors in the unperturbed medium:

$$\begin{pmatrix} \dfrac{\partial G^{s1}_{il}}{\partial s} & \dfrac{\partial G^{sf}_{il}}{\partial s} \\ \dfrac{\partial G^{sf}_{il}}{\partial s} & \dfrac{\partial G^{f2}_{il}}{\partial s} \end{pmatrix} = \begin{pmatrix} {}^0G^{s1}_{ij} & {}^0G^{sf}_{ij} \\ {}^0G^{sf}_{ij} & {}^0G^{f2}_{ij} \end{pmatrix} \cdot \begin{pmatrix} \dfrac{V^{s1}_{jk}}{\partial s} & \dfrac{V^{sf}_{jk}}{\partial s} \\ \dfrac{V^{sf}_{jk}}{\partial s} & \dfrac{V^{f2}_{jk}}{\partial s} \end{pmatrix} \cdot \begin{pmatrix} G^{s1}_{kl} & G^{sf}_{kl} \\ G^{sf}_{kl} & G^{f2}_{kl} \end{pmatrix}, \tag{65}$$

Under the assumtpion of week inhomogeneties, the Green's tensors in the perturbed medium can be replaced by the Green's tensors in the unperturbed medium:

$$\begin{pmatrix} \dfrac{\partial G^{s1}_{il}}{\partial s} & \dfrac{\partial G^{sf}_{il}}{\partial s} \\ \dfrac{\partial G^{sf}_{il}}{\partial s} & \dfrac{\partial G^{f2}_{il}}{\partial s} \end{pmatrix} \simeq \begin{pmatrix} {}^0G^{s1}_{ij} & {}^0G^{sf}_{ij} \\ {}^0G^{sf}_{ij} & {}^0G^{f2}_{ij} \end{pmatrix} \cdot \begin{pmatrix} \dfrac{V^{s1}_{jk}}{\partial s} & \dfrac{V^{sf}_{jk}}{\partial s} \\ \dfrac{V^{sf}_{jk}}{\partial s} & \dfrac{V^{f2}_{jk}}{\partial s} \end{pmatrix} \cdot \begin{pmatrix} {}^0G^{s1}_{kl} & {}^0G^{sf}_{kl} \\ {}^0G^{sf}_{kl} & {}^0G^{f2}_{kl} \end{pmatrix}, \tag{66}$$

Coupled Poroelastic Fr$\acute{e}$chet Derivatives

In this research, we defined 9 Fréchet derivatives matrices: **A**, **B**, **C**, **D**, **E**, **F**, **Q**, **T** and **Z** corresponding to the 9 poroelastic model parameters: λ_{dry},f, μ, ρ_{sat}, ρ_f, C, M, $\tilde{\rho}$ and m respectively. The parameter $f = \varphi^2 M$ is fluid/porosity term defined by Russell et al. (2011) for linearized poroelastic AVO analysis. In this research, we will discuss the wavefields response with respect to the perturbation of this parameter. The first element in the scattered wavefields matrix $\delta\mathbf{G}$ can be expressed as:

$$\begin{aligned} \delta G^{s1}_{xx} = {} & {}^0G^{s1}_{xx}V^{s10}_{xx}G^{s1}_{xx} + {}^0G^{s1}_{xx}V^{s10}_{xy}G^{s1}_{yx} + {}^0G^{s1}_{xx}V^{s10}_{xz}G^{s1}_{zx} + {}^0G^{s1}_{xy}V^{s10}_{yx}G^{s1}_{xx} + {}^0G^{s1}_{xy}V^{s10}_{yy}G^{s1}_{yx} \\ & + {}^0G^{s1}_{xy}V^{s10}_{yz}G^{s1}_{zx} + {}^0G^{s1}_{xz}V^{s10}_{zx}G^{s1}_{xx} + {}^0G^{s1}_{xz}V^{s10}_{zy}G^{s1}_{yx} + {}^0G^{s1}_{xz}V^{s10}_{zz}G^{s1}_{zx} + {}^0G^{sf}_{xx}V^{sf0}_{xx}G^{sf}_{xx} \\ & + {}^0G^{sf}_{xx}V^{sf0}_{xy}G^{sf}_{yx} + {}^0G^{sf}_{xx}V^{sf0}_{xz}G^{sf}_{zx} + {}^0G^{sf}_{xy}V^{sf0}_{yx}G^{sf}_{xx} + {}^0G^{sf}_{xy}V^{sf0}_{yy}G^{sf}_{yx} + {}^0G^{sf}_{xy}V^{sf0}_{yz}G^{sf}_{zx} \\ & + {}^0G^{sf}_{xz}V^{sf0}_{zx}G^{sf}_{xx} + {}^0G^{sf}_{xz}V^{sf0}_{zy}G^{sf}_{yx} + {}^0G^{sf}_{xz}V^{sf0}_{zz}G^{sf}_{zx} + {}^0G^{sf}_{xx}V^{sf0}_{xx}G^{s1}_{xx} + {}^0G^{sf}_{xx}V^{sf0}_{xy}G^{s1}_{yx} \\ & + {}^0G^{sf}_{xx}V^{sf0}_{xz}G^{s1}_{zx} + {}^0G^{sf}_{xy}V^{sf0}_{yx}G^{s1}_{xx} + {}^0G^{sf}_{xy}V^{sf0}_{yy}G^{s1}_{yx} + {}^0G^{sf}_{xy}V^{sf0}_{yz}G^{s1}_{zx} + {}^0G^{sf}_{xz}V^{sf0}_{zx}G^{s1}_{xx} \end{aligned} \tag{67}$$

$$+{}^0G^{sf}_{xz}V^{sf0}_{zy}{}^0G^{s1}_{yx}+{}^0G^{sf}_{xz}V^{sf0}_{zz}{}^0G^{s1}_{zx}+{}^0G^{sf}_{xx}V^{f20}_{xx}G^{sf}_{xx}+{}^0G^{sf}_{xx}V^{f20}_{xy}G^{sf}_{yx}+{}^0G^{sf}_{xx}V^{f20}_{xz}G^{sf}_{zx}$$
$$+{}^0G^{sf}_{xy}V^{f20}_{yx}G^{sf}_{xx}+{}^0G^{f2}_{xy}V^{sf0}_{yy}G^{sf}_{yx}+{}^0G^{sf}_{xy}V^{f20}_{yz}G^{sf}_{zx}+{}^0G^{sf}_{xz}V^{f20}_{zx}G^{sf}_{xx}+{}^0G^{sf}_{xz}V^{f20}_{zy}G^{sf}_{yx}$$
$$+{}^0G^{sf}_{xz}V^{f20}_{zz}G^{sf}_{zx}.$$

Similarly, the other elements in $\delta\mathbf{G}$ can be expressed as:

$$\begin{aligned}
\delta G^{s1}_{il} &= \sum_{i,j,k,l=x,y,z}\left({}^0G^{s1}_{ij}V^{s10}_{jk}G^{s1}_{kl}+{}^0G^{sf}_{ij}V^{sf0}_{jk}G^{sf}_{kl}+{}^0G^{sf}_{ij}V^{sf0}_{jk}G^{s1}_{kl}+{}^0G^{sf}_{ij}V^{f20}_{jk}G^{sf}_{kl}\right)\\
\delta G^{sf}_{il} &= \sum_{i,j,k,l=x,y,z}\left({}^0G^{s1}_{ij}V^{s10}_{jk}G^{sf}_{kl}+{}^0G^{s1}_{ij}V^{sf0}_{jk}G^{f2}_{kl}+{}^0G^{sf}_{ij}V^{sf0}_{jk}G^{sf}_{kl}+{}^0G^{sf}_{ij}V^{f20}_{jk}G^{f2}_{kl}\right)\\
\delta G^{f2}_{il} &= \sum_{i,j,k,l=x,y,z}\left({}^0G^{sf}_{ij}V^{f10}_{jk}G^{sf}_{kl}+{}^0G^{sf}_{ij}V^{sf0}_{jk}G^{f2}_{kl}+{}^0G^{f2}_{ij}V^{sf0}_{jk}G^{sf}_{kl}+{}^0G^{f2}_{ij}V^{f20}_{jk}G^{f2}_{kl}\right)
\end{aligned} \tag{68}$$

And to calculate these Fréchet derivatives matrices with respect to different parameters, firstly, we can derive the scattering potentials for different model parameters, which are listed in APPENDIX A. So, the Fréchet derivatives for different parameters become:

$$\mathbf{A}=\begin{pmatrix}\frac{\delta G^{s1}_{il}}{a_{\lambda_{dry}}} & \frac{\delta G^{sf}_{il}}{a_{\lambda_{dry}}}\\ \frac{\delta G^{sf}_{il}}{a_{\lambda_{dry}}} & \frac{\delta G^{f2}_{il}}{a_{\lambda_{dry}}}\end{pmatrix},\mathbf{B}=\begin{pmatrix}\frac{\delta G^{s1}_{il}}{a_f} & \frac{\delta G^{sf}_{il}}{a_f}\\ \frac{\delta G^{sf}_{il}}{a_f} & \frac{\delta G^{f2}_{il}}{a_f}\end{pmatrix},\mathbf{C}=\begin{pmatrix}\frac{\delta G^{s1}_{il}}{a_\mu} & \frac{\delta G^{sf}_{il}}{a_\mu}\\ \frac{\delta G^{sf}_{il}}{a_\mu} & \frac{\delta G^{f2}_{il}}{a_\mu}\end{pmatrix},\mathbf{D}=\begin{pmatrix}\frac{\delta G^{s1}_{il}}{a_{\rho_{sat}}} & \frac{\delta G^{sf}_{il}}{a_{\rho_{sat}}}\\ \frac{\delta G^{sf}_{il}}{a_{\rho_{sat}}} & \frac{\delta G^{f2}_{il}}{a_{\rho_{sat}}}\end{pmatrix}, \tag{69}$$

$$\mathbf{E}=\begin{pmatrix}\frac{\delta G^{s1}_{il}}{a_{\rho_f}} & \frac{\delta G^{sf}_{il}}{a_{\rho_f}}\\ \frac{\delta G^{sf}_{il}}{a_{\rho_f}} & \frac{\delta G^{f2}_{il}}{a_{\rho_f}}\end{pmatrix},\mathbf{F}=\begin{pmatrix}\frac{\delta G^{s1}_{il}}{a_C} & \frac{\delta G^{sf}_{il}}{a_C}\\ \frac{\delta G^{sf}_{il}}{a_C} & \frac{\delta G^{f2}_{il}}{a_C}\end{pmatrix},\mathbf{Q}=\begin{pmatrix}\frac{\delta G^{s1}_{il}}{a_M} & \frac{\delta G^{sf}_{il}}{a_M}\\ \frac{\delta G^{sf}_{il}}{a_M} & \frac{\delta G^{f2}_{il}}{a_M}\end{pmatrix},\mathbf{T}=\begin{pmatrix}\frac{\delta G^{s1}_{il}}{a_{\tilde\rho}} & \frac{\delta G^{sf}_{il}}{a_{\tilde\rho}}\\ \frac{\delta G^{sf}_{il}}{a_{\tilde\rho}} & \frac{\delta G^{f2}_{il}}{a_{\tilde\rho}}\end{pmatrix}, \tag{70}$$

$$\mathbf{Z}=\begin{pmatrix}\frac{\delta G^{s1}_{il}}{a_m} & \frac{\delta G^{sf}_{il}}{a_m}\\ \frac{\delta G^{sf}_{il}}{a_m} & \frac{\delta G^{f2}_{il}}{a_m}\end{pmatrix}. \tag{71}$$

The elements in matrix $\mathbf{A}$ can be expressed as:

$$\begin{aligned}
\frac{\delta G^{s1}_{il}}{a_{\lambda_{dry}}} &= \sum_{i,j,k,l=x,y,z}\left({}^0G^{s1}_{ij}\left(\frac{V^{s1}_{jk,\lambda_{dry}}}{a_{\lambda_{dry}}}\right){}^0G^{s1}_{kl}+{}^0G^{sf}_{ij}\left(\frac{V^{sf}_{jk,\lambda_{dry}}}{a_{\lambda_{dry}}}\right){}^0G^{sf}_{kl}+{}^0G^{sf}_{ij}\left(\frac{V^{sf}_{jk,\lambda_{dry}}}{a_{\lambda_{dry}}}\right){}^0G^{s1}_{kl}+{}^0G^{sf}_{ij}\left(\frac{V^{f2}_{jk,\lambda_{dry}}}{a_{\lambda_{dry}}}\right){}^0G^{sf}_{kl}\right),\\
\frac{\delta G^{sf}_{il}}{a_{\lambda_{dry}}} &= \sum_{i,j,k,l=x,y,z}\left({}^0G^{s1}_{ij}\left(\frac{V^{s1}_{jk,\lambda_{dry}}}{a_{\lambda_{dry}}}\right){}^0G^{sf}_{kl}+{}^0G^{s1}_{ij}\left(\frac{V^{sf}_{jk,\lambda_{dry}}}{a_{\lambda_{dry}}}\right){}^0G^{f2}_{kl}+{}^0G^{sf}_{ij}\left(\frac{V^{sf}_{jk,\lambda_{dry}}}{a_{\lambda_{dry}}}\right){}^0G^{sf}_{kl}+{}^0G^{sf}_{ij}\left(\frac{V^{f2}_{jk,\lambda_{dry}}}{a_{\lambda_{dry}}}\right){}^0G^{f2}_{kl}\right),\\
\frac{\delta G^{f2}_{il}}{a_{\lambda_{dry}}} &= \sum_{i,j,k,l=x,y,z}\left({}^0G^{sf}_{ij}\left(\frac{V^{f1}_{jk,\lambda_{dry}}}{a_{\lambda_{dry}}}\right){}^0G^{sf}_{kl}+{}^0G^{sf}_{ij}\left(\frac{V^{sf}_{jk,\lambda_{dry}}}{a_{\lambda_{dry}}}\right){}^0G^{f2}_{kl}+{}^0G^{f2}_{ij}\left(\frac{V^{sf}_{jk,\lambda_{dry}}}{a_{\lambda_{dry}}}\right){}^0G^{sf}_{kl}+{}^0G^{f2}_{ij}\left(\frac{V^{f2}_{jk,\lambda_{dry}}}{a_{\lambda_{dry}}}\right){}^0G^{f2}_{kl}\right).
\end{aligned} \tag{72}$$

We can substitute the scattering potentials $\boldsymbol{\mathcal{V}}_{\lambda_{dry}}$ for parameter λ_{dry} into the above equation and calculate the first element $\frac{\delta G^{s1}_{xx}}{a_{\lambda_{dry}}}$ in the Fréchet derivative matrix $\frac{\delta \mathbf{G}}{a_{\lambda_{dry}}}$:

$$\begin{aligned}\frac{\delta G^{s1}_{xx}}{a_{\lambda_{dry}}} &= {}^0G^{s1}_{xx}\left(\lambda^0_{dry}\partial_x\partial_x\right){}^0G^{s1}_{xx} + {}^0G^{s1}_{xx}\left(\lambda^0_{dry}\partial_x\partial_y\right){}^0G^{s1}_{yx} + {}^0G^{s1}_{xx}\left(\lambda^0_{dry}\partial_x\partial_z\right){}^0G^{s1}_{zx} \\ &+ {}^0G^{s1}_{xy}\left(\lambda^0_{dry}\partial_y\partial_x\right){}^0G^{s1}_{xx} + {}^0G^{s1}_{xy}\left(\lambda^0_{dry}\partial_y\partial_y\right){}^0G^{s1}_{yx} + {}^0G^{s1}_{xy}\left(\lambda^0_{dry}\partial_y\partial_z\right)^{s1}{}^0G^{s1}_{zx} \\ &+ {}^0G^{s1}_{xz}\left(\lambda^0_{dry}\partial_z\partial_x\right){}^0G^{s1}_{xx} + {}^0G^{s1}_{xz}\left(\lambda^0_{dry}\partial_z\partial_y\right){}^0G^{s1}_{yx} + {}^0G^{s1}_{xz}\left(\lambda^0_{dry}\partial_z\partial_z\right){}^0G^{s1}_{zx}.\end{aligned} \tag{73}$$

And the elements in matrices **B**, **C**, **D**, **E**, **F**, **Q**, **T** and **Z** can be expressed as:

$$\begin{aligned}\frac{\delta G^{s1}_{il}}{a_p} &= \sum_{i,j,k,l=x,y,z}\left({}^0G^{s1}_{ij}\left(\frac{V^{s1}_{jk,p}}{a_p}\right){}^0G^{s1}_{kl} + {}^0G^{sf}_{ij}\left(\frac{V^{sf}_{jk,p}}{a_p}\right){}^0G^{sf}_{kl} + {}^0G^{sf}_{ij}\left(\frac{V^{sf}_{jk,p}}{a_p}\right){}^0G^{s1}_{kl} + {}^0G^{sf}_{ij}\left(\frac{V^{f2}_{jk,p}}{a_p}\right){}^0G^{sf}_{kl}\right), \\ \frac{\delta G^{sf}_{il}}{a_p} &= \sum_{i,j,k,l=x,y,z}\left({}^0G^{s1}_{ij}\left(\frac{V^{s1}_{jk,p}}{a_p}\right){}^0G^{sf}_{kl} + {}^0G^{s1}_{ij}\left(\frac{V^{sf}_{jk,p}}{a_p}\right){}^0G^{f2}_{kl} + {}^0G^{sf}_{ij}\left(\frac{V^{sf}_{jk,p}}{a_p}\right){}^0G^{sf}_{kl} + {}^0G^{sf}_{ij}\left(\frac{V^{f2}_{jk,p}}{a_p}\right){}^0G^{f2}_{kl}\right), \\ \frac{\delta G^{f2}_{il}}{a_p} &= \sum_{i,j,k,l=x,y,z}\left({}^0G^{sf}_{ij}\left(\frac{V^{f1}_{jk,p}}{a_p}\right){}^0G^{sf}_{kl} + {}^0G^{sf}_{ij}\left(\frac{V^{sf}_{jk,p}}{a_p}\right){}^0G^{f2}_{kl} + {}^0G^{f2}_{ij}\left(\frac{V^{sf}_{jk,p}}{a_p}\right){}^0G^{sf}_{kl} + {}^0G^{f2}_{ij}\left(\frac{V^{f2}_{jk,p}}{a_p}\right){}^0G^{f2}_{kl}\right).\end{aligned} \tag{74}$$

where p indicate the model parameters: $p = \lambda_{dry}, f, \mu, \rho_{sat}, \rho_f, C, M, \tilde{\rho}$ and m and a_p indicate the model perturbations: $a_p = a_{\lambda_{dry}}, a_f, a_\mu, a_{\rho_{sat}}, a_{\rho_f}, a_C, a_M, a_{\tilde{\rho}}$ and a_m.

Coupled P-SV Fréchet Derivatives

According to Biot's theory, when wave propagating in poroelastic media, there are two kinds of P-wave and one kind of S-wave. As indicated by Fig.1, when the incident fast P-wave illuminates the scattering potential, the reflected fast P-wave, slow P-wave, SH wave and SV wave can be produced. The reference plane is defined by the incoming fast P-wave and outgoing fast P-wave. The SV wave and slow P-wave are lying within the reference plane. While the SH wave is normal to the reference plane. In low frequency limitation, the slow P-wave wavenumber is quite larger than fast P-wave wavenumber. So, the open angles θ_1 and θ_2 are different.

In this section, we follow Barros et al. (2008, 2010)'s method to analyze the P-SV Fréchet derivatives in poroelastic media. The governing equations for P-SV wave system are given in matrix form:

$$\boldsymbol{\mathcal{L}}^{PSV}\boldsymbol{U}^{PSV} = \boldsymbol{\mathcal{F}}^{PSV}, \tag{75}$$

where matrix $\boldsymbol{\mathcal{L}}^{PSV}$ is the differential operator:

$$\boldsymbol{\mathcal{L}}^{PSV} = \begin{pmatrix} \partial_z(\lambda_{dry}+f+2\mu)\partial_z + (\rho - p^2\mu)\omega^2 & -\omega p(\lambda_{dry}+f+\mu)\partial_z & \partial_z C\partial_z + \rho_f\omega^2 & -\partial_z C\omega p \\ (\lambda_{dry}+f+\mu)\omega p\partial_z & \partial_z\mu\partial_z + \rho_{sat}\omega^2 - p^2\omega^2(\lambda_{dry}+f+2\mu) & C\partial_z\omega p & -\omega^2p^2C \\ \partial_z C\partial_z + \rho_f\omega^2 & -C\omega p\partial_z & \partial_z M\partial_z + \tilde{\rho}\omega^2 & -M\omega p\partial_z \\ \omega pC\partial_z & \rho_f\omega^2 - p^2\omega^2 C & \omega pM\partial_z & \tilde{\rho}\omega^2 - Mp^2\omega^2 \end{pmatrix} \tag{76}$$

where $\tilde{\rho} = i\omega\frac{\eta}{k}$, p is the ray parameter and $f = \varphi^2 M$ in the fluid/porosity term (Russell et al., 2011). And $\boldsymbol{U}^{PSV}$ and $\boldsymbol{\mathcal{F}}^{PSV}$ are the displacements and source vectors respectively:

$$\boldsymbol{U}^{PSV} = \begin{pmatrix} \mathbf{U}_{sz} & \mathbf{U}_{sr} & \mathbf{U}_{fz} & \mathbf{U}_{fr} \end{pmatrix}^T, \boldsymbol{\mathcal{F}}^{PSV} = \begin{pmatrix} \mathbf{F}_{sz} & \mathbf{F}_{sr} & \mathbf{F}_{fz} & \mathbf{F}_{fr} \end{pmatrix}^T, \tag{77}$$

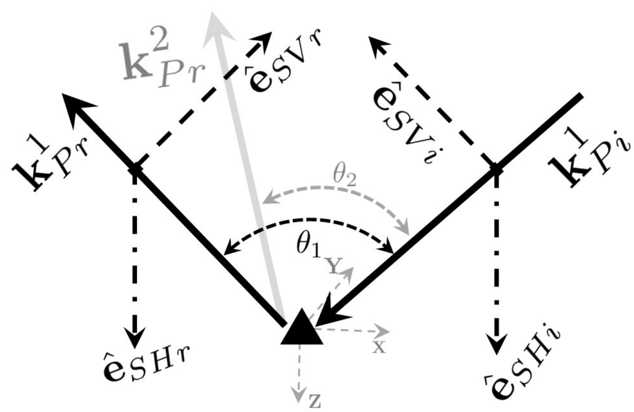

FIG. 1. Scattering scheme in poroelastic medium. $\mathbf{k}^1_{Pi}$ is the incident fast P-wave wavenumber. $\mathbf{k}^1_{Pr}$ and $\mathbf{k}^2_{Pr}$ are the reflected fast P-wave and slow P-wave wavenumbers respectively. $\hat{\mathbf{e}}_{SVi}$ and $\hat{\mathbf{e}}_{SHi}$ are the unit direction vectors for incident SV wave and SH wave. $\hat{\mathbf{e}}_{SVr}$ and $\hat{\mathbf{e}}_{SHr}$ are the unit direction vectors for reflected SV wave and SH wave. θ_1 is the open angle between the incident fast P-wave and reflected fast P-wave. θ_2 is the open angle between the incident fast P-wave and reflected slow P-wave.

$\mathbf{U}_{sz}$ and $\mathbf{U}_{sr}$ are the vertical and radial components of the solid displacements and $\mathbf{U}_{fz}$ and $\mathbf{U}_{fr}$ indicate the vertical and radial components of the relative fluid-to-solid displacements. And $\mathbf{F}_{sz}$, $\mathbf{F}_{sr}$, $\mathbf{F}_{fz}$ and $\mathbf{F}_{fr}$ are the corresponding sources. Here, we define the Fr$\acute{e}$chet derivatives $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$, $\mathbf{E}$, $\mathbf{F}$, $\mathbf{Q}$ and $\mathbf{T}$ for the model parameters of λ_{dry},f, μ, ρ_{sat}, ρ_f, C, M and $\tilde{\rho}$. So, the Fr$\acute{e}$chet derivatives $\mathbf{J}_{ij} = \frac{\delta \mathbf{U}_{ij}}{\delta p}$, $\mathbf{J} = \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}, \mathbf{E}, \mathbf{F}, \mathbf{Q}, \mathbf{T}$, $p = \lambda_{dry}, f, \mu, \rho_{sat}, \rho_f, C, M, \tilde{\rho}$; $i = s, f; j = z, r$. we can define the unperturbed wavefields and perturbated wavefields as:

$$\boldsymbol{U}_0^{PSV} = \begin{pmatrix} \mathbf{U}^0_{sz} & \mathbf{U}^0_{sr} & \mathbf{U}^0_{fz} & \mathbf{U}^0_{fr} \end{pmatrix}^T, \tag{78}$$

$$\boldsymbol{U}^{PSV} = \begin{pmatrix} \mathbf{U}_{sz} & \mathbf{U}_{sr} & \mathbf{U}_{fz} & \mathbf{U}_{fr} \end{pmatrix}^T,$$

So, the scattered wavefields corresponding to the small perturations of the model parameters can be writen as:

$$\delta \boldsymbol{U}^{PSV} = \boldsymbol{U}^{PSV} - \boldsymbol{U}_0^{PSV} = \begin{pmatrix} \delta\mathbf{U}_{sz} & \delta\mathbf{U}_{sr} & \delta\mathbf{U}_{fz} & \delta\mathbf{U}_{fr} \end{pmatrix}^T \tag{79}$$

We can also define the wave operators $\boldsymbol{\mathcal{L}}_0^{PSV}$ and $\boldsymbol{\mathcal{L}}^{PSV}$ in the unpertirbed and unperturbed medium:

$$\boldsymbol{\mathcal{L}}^{PSV} = \boldsymbol{\mathcal{L}}_0^{PSV} + \delta\boldsymbol{\mathcal{L}}^{PSV}, \tag{80}$$

where $\delta\boldsymbol{\mathcal{L}}^{PSV}$ can also be written as the scattering potentials $\boldsymbol{\mathcal{V}}^{PSV}$. The unperturbed and perturbed wave equations can be expressed as:

$$\boldsymbol{\mathcal{L}}_0^{PSV}\boldsymbol{U}_0^{PSV} = \boldsymbol{\mathcal{F}}^{\boldsymbol{PSV}}, \tag{81}$$

$$\left(\boldsymbol{\mathcal{L}}_0^{PSV} + \delta\boldsymbol{\mathcal{L}}^{PSV}\right)\left(\boldsymbol{U}_0^{PSV} + \delta\boldsymbol{U}^{PSV}\right) = \boldsymbol{\mathcal{F}}^{\boldsymbol{PSV}},$$

Based on the Born approximation, we can get:

$$\boldsymbol{\mathcal{L}}_0^{PSV}\delta\boldsymbol{U}^{PSV} = -\delta\boldsymbol{\mathcal{L}}^{PSV}\boldsymbol{U}^{PSV} \simeq -\delta\boldsymbol{\mathcal{L}}^{PSV}\boldsymbol{U}_0^{PSV} = \delta\boldsymbol{\mathcal{F}}^{\boldsymbol{PSV}}, \tag{82}$$

The right hand side of the above equation is always referred to as "scattered sources" or "secondary Born sources". The scattered wavefields $\delta\mathbf{U}$ due to the perturbations in the model parameters, can be interpreted as the wavefields generated by a set of secondary body forces propagating in the current, unperturbed medium $\boldsymbol{\mathcal{L}}_0$ (Dietrich and Kormendi, 1990). Isolating the scattered wavefields on the left hand side of the equation gives:

$$\delta\boldsymbol{U}^{PSV} = \int_{\mathcal{M}} \boldsymbol{\mathcal{G}}^{PSV}\delta\boldsymbol{F}^{PSV}dz \tag{83}$$

where $\boldsymbol{\mathcal{G}}^{PSV}$ and $\delta\boldsymbol{\mathcal{F}}^{PSV}$ are the Green's functions and scattered sources vectors:

$$\boldsymbol{\mathcal{G}}^{PSV} = \begin{pmatrix} G_{sz}^{sz}(z_g,z,\omega) & G_{sz}^{sr}(z_g,z,\omega) & G_{sz}^{fz}(z_g,z,\omega) & G_{sz}^{fr}(z_g,z,\omega) \\ G_{sr}^{sz}(z_g,z,\omega) & G_{sr}^{sr}(z_g,z,\omega) & G_{sr}^{fz}(z_g,z,\omega) & G_{sr}^{fr}(z_g,z,\omega) \\ G_{fz}^{sz}(z_g,z,\omega) & G_{fz}^{sr}(z_g,z,\omega) & G_{fz}^{fz}(z_g,z,\omega) & G_{fz}^{fr}(z_g,z,\omega) \\ G_{fr}^{sz}(z_g,z,\omega) & G_{fr}^{sr}(z_g,z,\omega) & G_{fr}^{fz}(z_g,z,\omega) & G_{fr}^{fr}(z_g,z,\omega) \end{pmatrix}, \tag{84}$$

$$\delta\boldsymbol{\mathcal{F}}^{PSV} = -\boldsymbol{\mathcal{V}}^{PSV}\boldsymbol{U}_0^{PSV} = \begin{pmatrix} \delta\mathbf{F}_{sz} & \delta\mathbf{F}_{sr} & \delta\mathbf{F}_{fz} & \delta\mathbf{F}_{fr} \end{pmatrix}^T, \tag{85}$$

where the P-SV scattering potentials $\boldsymbol{\mathcal{V}}^{PSV}$ are:

$$\boldsymbol{\mathcal{V}}^{PSV} = \begin{pmatrix} V_{11}^{PSV} & V_{12}^{PSV} & V_{13}^{PSV} & V_{14}^{PSV} \\ V_{21}^{PSV} & V_{22}^{PSV} & V_{23}^{PSV} & V_{24}^{PSV} \\ V_{31}^{PSV} & V_{32}^{PSV} & V_{33}^{PSV} & V_{34}^{PSV} \\ V_{41}^{PSV} & V_{42}^{PSV} & V_{43}^{PSV} & V_{44}^{PSV} \end{pmatrix}, \tag{86}$$

where

$$\begin{aligned}
V_{11}^{PSV} &= \partial_z(\lambda_{dry}^0 a_{\lambda_{dry}} + f^0 a_f + 2\mu^0 a_\mu)\partial_z + (\rho_{sat}^0 a_{\rho_{sat}} - p^2\mu^0 a_\mu)\omega^2, \\
V_{12}^{PSV} &= -\omega p(\lambda_{dry}^0 a_{\lambda_{dry}} + \mu^0 a_\mu)\partial_z, \\
V_{13}^{PSV} &= C^0\partial_z a_C\partial_z + \rho_f^0 a_{\rho_f}\omega^2, \\
V_{14}^{PSV} &= -\partial_z C\omega p, \\
V_{21}^{PSV} &= (\lambda_{dry}^0 a_{\lambda_{dry}} + f^0 a_f + \mu^0 a_\mu)\omega p\partial_z, \\
V_{22}^{PSV} &= \mu^0\partial_z a_\mu\partial_z + \rho_{sat}^0 a_{\rho_{sat}}\omega^2 - p^2\omega^2(\lambda_{dry}^0 a_{\lambda_{dry}} + f^0 a_f + 2\mu^0 a_\mu), \\
V_{23}^{PSV} &= -\partial_z C\omega p, \\
V_{24}^{PSV} &= -\omega^2 p^2 C^0 a_C, \\
V_{31}^{PSV} &= C^0\partial_z a_C\partial_z + \rho_f^0 a_{\rho_f}\omega^2, \\
V_{32}^{PSV} &= -C^0 a_C\omega p\partial_z, \\
V_{33}^{PSV} &= M^0\partial_z a_M\partial_z + \tilde{\rho}^0 a_{\tilde{\rho}}\omega^2, \\
V_{34}^{PSV} &= -M^0 a_M\omega p\partial_z, \\
V_{41}^{PSV} &= \omega p C^0 a_C\partial_z, \\
V_{42}^{PSV} &= \rho_f^0 a_{\rho_f}\omega^2 - p^2\omega^2 C^0 a_C, \\
V_{43}^{PSV} &= \omega p M^0 a_M\partial_z, \\
V_{44}^{PSV} &= \tilde{\rho}^0 a_{\tilde{\rho}}\omega^2 - M^0 a_M p^2\omega^2.
\end{aligned} \tag{87}$$

So, the P-SV Fréchet derivatives can be expressed in a integral representation:

$$\frac{\delta \boldsymbol{U}^{PSV}}{\delta P} = -\int_{\mathcal{M}} \boldsymbol{\mathcal{G}}^{PSV} \left(\frac{\boldsymbol{\mathcal{V}}^{PSV}}{\delta P} \boldsymbol{U}_0^{PSV} \right) dz, \tag{88}$$

Fréchet derivatives with respect to λ_{sat}

The Fréchet derivatives with respect to λ_{dry} mean the wavefields response corresponding the change in the model parameter λ_{dry}. Firstly, we can obtain the P-SV scattering potential $\boldsymbol{\mathcal{V}}_{\lambda_{dry}}^{PSV}$ for λ_{dry} by setting all of the other model perturbations $a_{\lambda_{dry}}$, a_{μ}, $a_{\rho_{sat}}$, a_{ρ_f}, a_C, a_M, and $a_{\tilde{\rho}}$ as 0:

$$\boldsymbol{\mathcal{V}}_{\lambda_{sat}}^{PSV} = \begin{pmatrix} \lambda_{dry}^0 \partial_z a_{\lambda_{dry}} \partial_z & -\omega p \lambda_{dry}^0 a_{\lambda_{dry}} \partial_z & 0 & 0 \\ \lambda_{dry}^0 a_{\lambda_{dry}} \omega p \partial_z & -p^2 \omega^2 \lambda_{dry}^0 a_{\lambda_{dry}} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{89}$$

Substituting $\boldsymbol{\mathcal{V}}_{\lambda_{dry}}^{PSV}$ into equation (96), we can get the Fréchet derivatives for parameter λ_{dry}:

$$\mathbf{A} = \frac{\delta \boldsymbol{U}^{PSV}}{a_{\lambda_{dry}}} = -\int_{\mathcal{M}} \boldsymbol{\mathcal{G}}^{PSV} \left(\frac{\delta \boldsymbol{\mathcal{L}}_{\lambda_{dry}}^{PSV}}{a_{\lambda_{dry}}} \boldsymbol{U}_0^{PSV} \right) dz, \tag{90}$$

$$\mathbf{A} = \begin{pmatrix} \frac{\delta \mathbf{U}_{sz}}{a_{\lambda_{dry}}} & \frac{\delta \mathbf{U}_{sr}}{a_{\lambda_{dry}}} & \frac{\delta \mathbf{U}_{fz}}{a_{\lambda_{dry}}} & \frac{\delta \mathbf{U}_{fr}}{a_{\lambda_{dry}}} \end{pmatrix}^T \tag{91}$$

So, the Fréchet derivatives for parameter λ_{dry} can be expressed using the displacements and Green's functions:

$$\frac{\delta \mathbf{U}_{ij}}{a_{\lambda_{dry}}} = -\lambda_{dry}^0 \partial_z \mathbf{U}_{sz}^0 G_{ij}^{sz} \partial_z + \lambda_{dry}^0 \omega p \mathbf{U}_{sr}^0 \partial_z G_{ij}^{sz} + \lambda_{dry}^0 \omega p \partial_z G_{ij}^{sr} \mathbf{U}_{sz}^0 - \lambda_{dry}^0 \omega^2 p^2 \mathbf{U}_{sr}^0 G_{ij}^{sr}, i = s, f; j = z, r. \tag{92}$$

Fréchet derivatives with respect to f

The porosity/fluid term $f = \varphi^2 M$ was involved by (Russell et al., 2011) for linearized poroelastic AVO analysis. In this research, we derived the Fréchet derivative matrix with respect to f:

$$\boldsymbol{\mathcal{V}}_f^{PSV} = \begin{pmatrix} \partial_z f^0 a_f \partial_z & -\omega p f^0 a_f \partial_z & 0 & 0 \\ f^0 a_f \omega p \partial_z & -p^2 \omega^2 f^0 a_f & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{93}$$

Substituting $\boldsymbol{\mathcal{V}}_f^{PSV}$ into equation (96), we can get the Fréchet derivatives for parameter f:

$$\mathbf{A} = \frac{\delta \boldsymbol{U}^{PSV}}{a_f} = -\int_{\mathcal{M}} \boldsymbol{\mathcal{G}}^{PSV} \left(\frac{\delta \boldsymbol{\mathcal{L}}_f^{PSV}}{a_f} \boldsymbol{U}_0^{PSV} \right) dz, \tag{94}$$

$$\mathbf{A} = \begin{pmatrix} \frac{\delta \mathbf{U}_{sz}}{a_f} & \frac{\delta \mathbf{U}_{sr}}{a_f} & \frac{\delta \mathbf{U}_{fz}}{a_f} & \frac{\delta \mathbf{U}_{fr}}{a_f} \end{pmatrix}^T \tag{95}$$

So, the Fréchet derivatives for parameter f can be expressed using the displacements and Green's functions:

$$\frac{\delta \mathbf{U}_{ij}}{a_f} = -f^0\partial_z \mathbf{U}^0_{sz} G^{sz}_{ij}\partial_z + f^0\omega p \mathbf{U}^0_{sr}\partial_z G^{sz}_{ij} + f^0\omega p \partial_z G^{sr}_{ij}\mathbf{U}^0_{sz} - f^0\omega^2 p^2 \mathbf{U}^0_{sr} G^{sr}_{ij}, i = s, f; j = z, r. \tag{96}$$

Fréchet derivatives with respect to μ

To obtain the Fréchet derivatives with respect to parameter μ, we can calculate the scattering potentials for parameter μ firstly:

$$\boldsymbol{\mathcal{V}}^{PSV}_{\mu} = \begin{pmatrix} 2\partial_z\mu^0 a_\mu\partial_z - p^2\mu^0 a_\mu\omega^2 & -\omega p\mu^0 a_\mu\partial_z & 0 & 0 \\ \mu^0 a_\mu\omega p\partial_z & \partial_z\mu^0 a_\mu\partial_z - 2p^2\omega^2\mu^0 a_\mu & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{97}$$

The Fréchet derivatives vector **B** for parameter μ:

$$\mathbf{B} = \frac{\delta \boldsymbol{U}^{PSV}}{a_\mu} = -\int_{\mathcal{M}} \boldsymbol{\mathcal{G}}^{PSV}\left(\frac{\boldsymbol{\mathcal{V}}^{PSV}_{\mu}}{a_\mu}\boldsymbol{U}^{PSV}_0\right) dz, \tag{98}$$

$$\mathbf{B} = \begin{pmatrix} \frac{\delta \mathbf{U}_{sz}}{a_\mu} & \frac{\delta \mathbf{U}_{sr}}{a_\mu} & \frac{\delta \mathbf{U}_{fz}}{a_\mu} & \frac{\delta \mathbf{U}_{fr}}{a_\mu} \end{pmatrix}^T \tag{99}$$

And Fréchet derivatives with respect to parameter μ can be expressed as:

$$\frac{\delta \mathbf{U}_{ij}}{a_\mu} = -\left(2\partial_z^2 - p^2\omega^2 + \omega p\partial_z\right)\mu^0\mathbf{U}^0_{sz}G^{sz}_{ij} + \left(\omega p\partial_z + \partial_z^2 - 2p^2\omega^2\right)\mu^0\mathbf{U}^0_{sr}G^{sr}_{ij}, i = s, f; j = z, r. \tag{100}$$

Fréchet derivatives with respect to ρ_{sat}

The scattering potentials for parameter ρ_{sat} can be written as:

$$\boldsymbol{\mathcal{V}}^{PSV}_{\rho_{sat}} = \begin{pmatrix} \rho^0_{sat}a_{\rho_{sat}}\omega^2 & 0 & 0 & 0 \\ 0 & \rho^0_{sat}a_{\rho_{sat}}\omega^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{101}$$

And the corresponding Fréchet derivatives with respect to parameter ρ_{sat} can be expressed as:

$$\mathbf{C} = \frac{\delta \boldsymbol{U}^{PSV}}{a_{\rho_{sat}}} = -\int_{\mathcal{M}} \boldsymbol{\mathcal{G}}^{PSV}\left(\frac{\delta\boldsymbol{\mathcal{L}}^{PSV}_{\rho_{sat}}}{a_{\rho_{sat}}}\boldsymbol{U}^{PSV}_0\right) dz, \tag{102}$$

$$\mathbf{C} = \begin{pmatrix} \frac{\delta \mathbf{U}_{sz}}{\delta\rho_{sat}} & \frac{\delta \mathbf{U}_{sr}}{a_{\rho_{sat}}} & \frac{\delta \mathbf{U}_{fz}}{a_{\rho_{sat}}} & \frac{\delta \mathbf{U}_{fr}}{a_{\rho_{sat}}} \end{pmatrix}^T \tag{103}$$

$$\frac{\delta \mathbf{U}_{ij}}{a_{\rho_{sat}}} = \rho^0_{sat}\omega^2\mathbf{U}^0_{sz}G^{sz}_{ij} + \rho^0_{sat}\omega^2\mathbf{U}^0_{sr}G^{sr}_{ij}, i = s, f; j = z, r. \tag{104}$$

Fréchet derivatives with respect to ρ_f

The scattering potentials for parameter ρ_f are:

$$\boldsymbol{\mathcal{V}}_{\rho_f}^{PSV} = \begin{pmatrix} 0 & 0 & a_{\rho_f}\omega^2 & 0 \\ 0 & 0 & 0 & 0 \\ a_{\rho_f}\omega^2 & 0 & 0 & 0 \\ 0 & a_{\rho_f}\omega^2 & 0 & 0 \end{pmatrix} \tag{105}$$

And the corresponding Fréchet derivatives $\mathbf{D}$ are:

$$\mathbf{D} = \frac{\delta \boldsymbol{U}^{PSV}}{a_{\rho_f}} = -\int_{\mathcal{M}} \boldsymbol{\mathcal{G}}^{PSV} \left(\frac{\delta \boldsymbol{\mathcal{L}}_{\rho_f}^{PSV}}{\delta \rho_f} \boldsymbol{U}_0^{PSV} \right) dz, \tag{106}$$

$$\mathbf{D} = \begin{pmatrix} \frac{\delta \mathbf{U}_{sz}}{a_{\rho_f}} & \frac{\delta \mathbf{U}_{sr}}{a_{\rho_f}} & \frac{\delta \mathbf{U}_{fz}}{a_{\rho_f}} & \frac{\delta \mathbf{U}_{fr}}{a_{\rho_f}} \end{pmatrix}^T \tag{107}$$

$$\frac{\delta \mathbf{U}_{ij}}{a_{\rho_f}} = -\rho_f^0 \omega^2 \mathbf{U}_{fz}^0 G_{ij}^{sz} - \rho_f^0 \omega^2 \mathbf{U}_{sz}^0 G_{ij}^{fz} - \rho_f^0 \omega^2 \mathbf{U}_{sr}^0 G_{ij}^{fr}, i = s, f; j = z, r. \tag{108}$$

Fréchet derivatives with respect to C

The scattering potentials for parameter C are:

$$\boldsymbol{\mathcal{V}}_{C}^{PSV} = \begin{pmatrix} 0 & 0 & \partial_z C^0 a_C \partial_z & -\partial_z C^0 a_C \omega p \\ 0 & 0 & C^0 a_C \partial_z \omega p & -\omega^2 p^2 C^0 a_c \\ \partial_z C^0 a_C \partial_z & -C^0 a_C \omega p \partial_z & 0 & 0 \\ \omega p C^0 a_C \partial_z & -p^2 \omega^2 C^0 a_C & 0 & 0 \end{pmatrix} \tag{109}$$

And the corresponding Fréchet derivatives can be expressed as:

$$\mathbf{E} = \frac{\delta \boldsymbol{U}^{PSV}}{a_C} = -\int_{\mathcal{M}} \boldsymbol{\mathcal{G}}^{PSV} \left(\frac{\delta \boldsymbol{\mathcal{L}}_{C}^{PSV}}{a_C} \boldsymbol{U}_0^{PSV} \right) dz, \tag{110}$$

$$\mathbf{E} = \begin{pmatrix} \frac{\delta \mathbf{U}_{sz}}{a_C} & \frac{\delta \mathbf{U}_{sr}}{a_C} & \frac{\delta \mathbf{U}_{fz}}{a_C} & \frac{\delta \mathbf{U}_{fr}}{a_C} \end{pmatrix}^T \tag{111}$$

$$\begin{aligned} \frac{\delta \mathbf{U}_{ij}}{a_C} = {} & C^0 \left(\partial_z^2 \mathbf{U}_{fz}^0 G_{ij}^{sz} - \partial_z \omega p \mathbf{U}_{fr}^0 G_{ij}^{sz} - \partial_z \omega p \mathbf{U}_{fz}^0 G_{ij}^{sr} - \omega^2 p^2 \mathbf{U}_{fr}^0 G_{ij}^{sr} + \partial^2 \mathbf{U}_{sz}^0 G_{ij}^{fz} \right) \\ & - C^0 \left(\omega p \partial_z \mathbf{U}_{sr}^0 G_{ij}^{fz} + \omega p \partial_z \mathbf{U}_{sz}^0 G_{ij}^{fr} - p^2 \omega^2 \mathbf{U}_{sr}^0 G_{ij}^{fr} \right), i = s, f; j = z, r. \end{aligned} \tag{112}$$

Fréchet derivatives with respect to M

The scattering potentials for parameter M are:

$$\boldsymbol{\mathcal{V}}_{M}^{PSV} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \partial_z M^0 a_M \partial_z & -M^0 a_M \omega p \partial_z \\ 0 & 0 & \omega p M^0 a_M \partial_z & -M^0 a_M p^2 \omega^2 \end{pmatrix} \tag{113}$$

And the Fr$\acute{e}$chet derivatives are:

$$\mathbf{F} = \frac{\delta \boldsymbol{U}^{PSV}}{a_M} = -\int_{\mathcal{M}} \boldsymbol{\mathcal{G}}^{PSV} \left(\frac{\delta \boldsymbol{\mathcal{L}}_M^{PSV}}{aM} \boldsymbol{U}_0^{PSV} \right) dz, \tag{114}$$

$$\mathbf{F} = \begin{pmatrix} \frac{\delta \mathbf{U}_{sz}}{a_M} & \frac{\delta \mathbf{U}_{sr}}{a_M} & \frac{\delta \mathbf{U}_{fz}}{a_M} & \frac{\delta \mathbf{U}_{fr}}{a_M} \end{pmatrix}^T \tag{115}$$

$$\frac{\delta \mathbf{U}_{ij}}{\delta M} = M^0 \partial_z^2 \mathbf{U}_{fz}^0 G_{ij}^{fz} - M^0 \omega p \partial_z \mathbf{U}_{fr}^0 G_{ij}^{fz} + M^0 \omega p \partial_z \mathbf{U}_{fz}^0 G_{ij}^{fr} - M^0 p^2 \omega^2 \mathbf{U}_{fr}^0 G_{ij}^{fr}, i = s, f; j = z, r. \tag{116}$$

Fr$\acute{e}$chet derivatives with respect to $\tilde{\rho}$

The scattering potentials for parameter $\tilde{\rho}$ are:

$$\boldsymbol{\mathcal{V}}_{\tilde{\rho}}^{PSV} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \tilde{\rho}^0 a_{\tilde{\rho}} \omega^2 & 0 \\ 0 & 0 & 0 & \tilde{\rho}^0 a_{\tilde{\rho}} \omega^2 \end{pmatrix} \tag{117}$$

And the corresponding Fr$\acute{e}$chet derivatives are:

$$\mathbf{I} = \frac{\delta \boldsymbol{U}^{PSV}}{a_{\tilde{\rho}}} = -\int_{\mathcal{M}} \boldsymbol{\mathcal{G}}^{PSV} \left(\frac{\delta \boldsymbol{\mathcal{L}}_{\tilde{\rho}}^{PSV}}{\delta \tilde{\rho}} \boldsymbol{U}_0^{PSV} \right) dz, \tag{118}$$

$$\mathbf{I} = \begin{pmatrix} \frac{\delta \mathbf{U}_{sz}}{a_{\tilde{\rho}}} & \frac{\delta \mathbf{U}_{sr}}{a_{\tilde{\rho}}} & \frac{\delta \mathbf{U}_{fz}}{a_{\tilde{\rho}}} & \frac{\delta \mathbf{U}_{fr}}{a_{\tilde{\rho}}} \end{pmatrix}^T \tag{119}$$

$$\frac{\delta \mathbf{U}_{ij}}{a_{\tilde{\rho}}} = \tilde{\rho}^0 \omega^2 \mathbf{U}_{fz}^0 G_{ij}^{fz} + \tilde{\rho}^0 \omega^2 \mathbf{U}_{fr}^0 G_{ij}^{fr}, i = s, f; j = z, r. \tag{120}$$

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APPENDIX A: SCATTERING POTENTIALS FOR DIFFERENT POROELASTIC PARAMETERS

The whole poroleastic scattering potentials are listed from equations (35) to (39). And the poroelastic scattering potential for model parameter p can be obtained by keeping model perturbation a_p and setting other model perturbations as 0. So, the poroelastic scattering potential $\boldsymbol{\mathcal{V}}_{\rho_{sat}}$ for model parameter ρ_{sat} can be obtained by setting $a_f = 0$, $a_{\lambda_{dry}} = 0$, $a_\mu = 0$, $a_{\rho_f} = 0$, $a_C = 0$, $a_m = 0$, $a_M = 0$ and $a_{\tilde{\rho}} = 0$:

$$\boldsymbol{\mathcal{V}}_{\rho_{sat}} = \begin{pmatrix} \delta_{ij} \rho_{sat}^0 a_\rho \omega^2 & 0 \\ 0 & 0 \end{pmatrix}, i, j = x, y, z, \tag{121}$$

This means that the whole scattering potential can be written as a summation of the scattering potentials for different poroelastic model parameters:

$$\boldsymbol{\mathcal{V}} = \boldsymbol{\mathcal{V}}_{\lambda_{dry}} + \boldsymbol{\mathcal{V}}_{f} + \boldsymbol{\mathcal{V}}_{\mu} + \boldsymbol{\mathcal{V}}_{\rho_{sat}} + \boldsymbol{\mathcal{V}}_{\rho_f} + \boldsymbol{\mathcal{V}}_{C} + \boldsymbol{\mathcal{V}}_{M} + \boldsymbol{\mathcal{V}}_{\tilde{\rho}} + \boldsymbol{\mathcal{V}}_{m}, \tag{122}$$

Similarly, we can get the scattering potentials $\boldsymbol{\mathcal{V}}_{\lambda_{dry}}$,$\boldsymbol{\mathcal{V}}_{f}$, $\boldsymbol{\mathcal{V}}_{\mu}$, $\boldsymbol{\mathcal{V}}_{\rho_f}$, $\boldsymbol{\mathcal{V}}_{C}$, $\boldsymbol{\mathcal{V}}_{M}$, $\boldsymbol{\mathcal{V}}_{\tilde{\rho}}$ and $\boldsymbol{\mathcal{V}}_{m}$ for other model parameters respectively:

$$\begin{aligned}
\boldsymbol{\mathcal{V}}_{\lambda_{dry}} &= \begin{pmatrix} \lambda_{dry}^0 \partial_i a_{\lambda_{dry}} \partial_j & 0 \\ 0 & 0 \end{pmatrix}, i,j = x,y,z, \\
\boldsymbol{\mathcal{V}}_{f} &= \begin{pmatrix} f^0 \partial_i a_f \partial_j & 0 \\ 0 & 0 \end{pmatrix}, i,j = x,y,z, \\
\boldsymbol{\mathcal{V}}_{\mu} &= \begin{pmatrix} \rho_{sat}^0 (\beta_{sat}^0)^2 \partial_j a_\mu \partial_i + \delta_{ij} \rho_{sat}^0 (\beta_{sat}^0)^2 \sum_{j \neq i} \partial_j a_\mu \partial_j & 0 \\ 0 & 0 \end{pmatrix}, i,j = x,y,z, \\
\boldsymbol{\mathcal{V}}_{\rho_f} &= \begin{pmatrix} 0 & \delta_{ij} \rho_f^0 a_{\rho_f} \omega^2 \\ \delta_{ij} \rho_f^0 a_{\rho_f} \omega^2 & 0 \end{pmatrix}, i,j = x,y,z, \\
\boldsymbol{\mathcal{V}}_{C} &= \begin{pmatrix} 0 & C_0 \partial_i a_c \partial_j \\ C_0 \partial_i a_c \partial_j & 0 \end{pmatrix}, i,j = x,y,z, \\
\boldsymbol{\mathcal{V}}_{M} &= \begin{pmatrix} 0 & 0 \\ 0 & -M_0 \partial_i a_M \partial_j \end{pmatrix}, i,j = x,y,z, \\
\boldsymbol{\mathcal{V}}_{\tilde{\rho}} &= \begin{pmatrix} 0 & 0 \\ 0 & \tilde{\rho}^0 a_{\tilde{\rho}} \omega^2 \end{pmatrix}, \\
\boldsymbol{\mathcal{V}}_{m} &= \begin{pmatrix} 0 & 0 \\ 0 & -m_0 a_m \omega^2 \end{pmatrix}.
\end{aligned} \tag{123}$$

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