Absorbing boundaries in acoustic wave finite difference analogues

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ABSTRACT

The acoustic wave equation is considered using a combination of finite integral transforms and finite differences in the solution method. This approach is known as the pseudo – spectral method or Alekseev – Mikhailenko Method (AMM) after the Russian researchers who spent decades pursuing this avenue of solution which has resulted in numerous dozens of related papers in the literature. One aspect of this solution method that is often referred to but has received little attention in the literature is the introduction of absorbing boundaries. Using the simple acoustic wave equation in this solution type, the algorithm developed by Clayton and Engquist (1977) is incorporated into the transformed wave equation to produce a manner of introducing an absorbing at the model bottom.

INTRODUCTION

The acoustic wave equation in a radial symmetric medium is employed in the derivation of absorbing boundary conditions at the fictitious boundary at the model bottom. Before this development is investigated, the wave equation has its radial dependence removed through the use of a finite Hankel transform. The resultant equation is then treated in a similar manner to that used in the paper of Clayton and Engquist (1977) where paraxial approximations to the two dimensional acoustic wave equation are derived.

The method of employing finite integral transforms to remove one or more spatial dimensions so that a hyperbolic (wave) system of equations is reduced to a finite difference problem in depth and time is most often referred to as the pseudo-spectral method. However, due to the extensive work done in this area by B.G. Mikhailenko and A.S. Alekseev, it is sometimes referred to, in seismic applications, as the Alekseev-Mikhailenko Method (AMM), (Alekseev and Mikhailenko, 1980). It falls within the genetic class of pseudo-spectral methods, but is possibly more formal and rigorous in its development. However, much of their work is relatively physically inaccessible and a considerable number of the more significant contributions are in Russian. Other works of interest in this area are Gazdag (1973), Gazdag (1981) and Kosloff and Baysal (1982). Rather than attempt to solve the problem of wave propagation for the acoustic equation, a finite integral transform will be applied and then a plane wave solution will be used to remove the depth spatial dimension (z) and time. The resulting expression will then be manipulated to obtain parabolic equations, similar to those used by Clayton and Engquist (1977), in an effort to introduce an absorbing boundary in the transformed equation.

ACOUSTIC WAVE EQUATION

The well-known acoustic partial differential wave equation has the form

$$\rho v_P^2 \nabla^2 \psi - \rho \frac{\partial^2 \psi}{dt^2} = \delta(\mathbf{x}) f(t). \tag{1}$$

where the point source term (usually explosive) is on the right hand side and is solved with initial conditions

$$\psi(\mathbf{x},t)\Big|_{t=0} = \frac{\partial \psi}{dt}(\mathbf{x},t)\Big|_{t=0} = 0. \tag{2}$$

Depth (z) is assumed to be positive downwards and radial symmetry has been assumed $(2.5D)$. For the homogeneous case with no dependence of the media parameters on spatial variables, and no source term, the equation of motion for this case is given by

$$\rho v_P^2 \left(\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial \psi}{\partial r} \right) + \frac{\partial^2 \psi}{\partial z^2} \right) - \rho \frac{\partial^2 \psi}{dt^2} = 0. \tag{3}$$

First apply a finite Hankel transform on the interval $(0,a)$ where $r = a$ is a fictitiously introduced boundary in the radial direction where $J_0(k_j a) = 0$ is a transcendental equation that is solved numerically and the boundary condition $\psi_{r=a} = 0$, which implies that the introduced boundary at $r = a$ is perfectly reflecting. The finite Hankel transform is defined as

$$\psi(k_j, z, t) = \int_0^a J_0(k_j r)\psi(r,z,t)\, r dr \tag{4}$$

where, as previously noted, the k_j are obtained from the solution of the transcendental equation $J_0(k_j a) = 0$. The inverse of (4) is

$$\psi(r,z,t) = \frac{2}{a^2}\sum_{i=1}^{\infty} \frac{\psi(k_i, z, t) J_1(k_i r)}{\left[J_0(k_i a)\right]^2} \tag{5}$$

It is clear that some finite upper bound must be determined, usually dictated by the frequency spectrum of the source wavelet, for the inverse series. The result of this transformation is

$$-\rho v_P^2 k_j^2 \psi + \rho v_P^2 \frac{\partial^2 \psi}{\partial z^2} - \rho \frac{\partial^2 \psi}{dt^2} = 0. \tag{6}$$

If Fourier spatial and temporal transforms with respect to depth *(z)* and time (*t*) are applied $\left[\psi = \psi \exp(ik_z z - i\omega t) \right]$ then

$$\left[-\rho v_P^2 k_j^2 \psi + \rho v_P^2 (ik_z)^2 - \rho(-i\omega)^2 \right]\psi = 0. \tag{7}$$

Under the assumption, $\psi \neq 0$, it follows that

$$-\rho v_P^2 k_j^2 + \rho v_P^2 (ik_z)^2 - \rho(-i\omega)^2 = 0 \tag{8}$$

or equivalently

$$v_P^2\left(ik_z\right)^2=v_P^2k_j^2+\left(-i\omega\right)^2. \tag{9}$$

Rearranging and taking the downward propagating "+" sign produces

$$\frac{\left(ik_z\right)}{\left(-i\omega\right)}=\frac{1}{v_p}\left(1+\frac{v_P^2k_j^2}{\left(-i\omega\right)^2}\right)^{1/2}. \tag{10}$$

Approximating the $\left(\circ\right)^{1/2}$ radical under the assumption that $v_P^2k_j^2/\left(-i\omega\right)^2$ is $\ll 1$, has

$$\left(ik_z\right)\left(-i\omega\right)\approx\frac{\left(-i\omega\right)^2}{v_p}+\frac{v_pk_j^2}{2}. \tag{11}$$

It is to be reiterated that the above is truly only valid if $\left|\frac{v_P^2k_j^2}{\left(-i\omega\right)^2}\right|\ll 1$.

Rearranging equation (11) results in

$$\left(ik_z\right)\left(-i\omega\right)-\frac{\left(-i\omega\right)^2}{v_p}-\frac{v_pk_j^2}{2}=0 \tag{12}$$

With $\left(ik_z\right)\rightarrow\partial/\partial z$ and $\left(-i\omega\right)\rightarrow\partial/\partial t$ and applying these operations to ψ yields

$$\frac{\partial^2\psi}{\partial z\partial t}-\frac{1}{v_p}\frac{\partial^2\psi}{\partial t^2}-\frac{v_pk_j^2}{2}\psi=0 \tag{13}$$

An explicit finite difference analogue for (13) is obtained by standard second order methods as

$$\left(\psi_n^{m+1}-\psi_{n-1}^{m+1}-\psi_n^m+\psi_{n-1}^m\right)-\frac{\Delta z}{v_p\Delta t}\left(\psi_n^{m+1}-2\psi_n^m+\psi_n^{m-1}\right)-\frac{\Delta t\Delta z v_pk_j^2}{2}\psi_n^m=0 \tag{14}$$

with Δt and Δz being the time and depth increments with standard stability conditions assumed for finite difference analogues of order $\left[O\left(\Delta z\right)^2,O\left(\Delta t\right)^2\right]$ Rearranging results in

$$\left(1+\frac{\Delta z}{v_p\Delta t}\right)\psi_n^{m+1}=\psi_{n-1}^{m+1}+\psi_n^m-\psi_{n-1}^m-\frac{\Delta z}{v_p\Delta t}\left(2\psi_n^m-\psi_n^{m-1}\right)-\frac{\Delta t\Delta z v_pk_j^2}{2}\psi_n^m \tag{15}$$

Further simplification of (15) may be done but it is more useful in its present form. Theoretically, equation (15) provides an absorbing boundary at the model bottom for the problem considered here. However, much computed information remains unused in (15). The accepted manner to proceed in this problem is to utilize the (known) full waveform solution at the second last point in the vertical grid (point *K-1*) together with the paraxial $\left(15^\circ\right)$ approximation for the K^{th} point. This is discussed in more detail in Clayton and Engquist (1977). Modifying their absorbing condition at the model bottom for the acoustic wave equation which has undergone a finite transform of some type results in

$$D_-^z D_0^t \psi_K^m - \frac{v_K}{2} D_+^t D_-^t \left(\psi_K^m + \psi_{K-1}^m\right) - \frac{k_j^2}{4}\left(\psi_{K-1}^{m+1} + \psi_K^{m-1}\right) = 0 \tag{16}$$

where the $D_{+/-/0}^q$ are the forward, backward and center finite difference analogues with respect to the variable $q\left(q = z, t\right)$. The operators D_+^q , D_-^q and D_0^q are the forward, backward and center difference finite difference analogues with respect to the variable q and are defined as

$$\text{Forward}: \quad D_+^z G_k^n \approx \left(G_{k+1}^n - G_k^n\right) / \Delta z \tag{17}$$

$$\text{Backward}: \quad D_-^z G_k^n \approx \left(G_k^n - G_{k-1}^n\right) / \Delta z \tag{18}$$

$$\text{Center}: \quad D_0^t G_k^n \approx \left(G_k^{n+1} - G_k^{n-1}\right) / \left(2\Delta t\right)$$

actually (19)

$$D_0^t G_k^n \approx \left(G_k^{n+1/2} - G_k^{n-1/2}\right) / \Delta t$$

The introduction of (17) – (19) into (16) produces the desired absorbing boundary conditions at the model bottom.

HIGHER ORDER APPROXIMATIONS

A Padé approximation to equation (10) can be assumed to be of the form

$$\eta = \frac{v_p\left(ik_z\right)}{\left(-i\omega\right)} = \left(1 + \frac{v_P^2 k_j^2}{\left(-i\omega\right)^2}\right)^{1/2} \approx \frac{1 + a_2\xi^2}{1 + b_2\xi^2}. \tag{20}$$

for some a_2 and b_2 which are constants to be determined and where it has been implied that $\xi = \frac{v_P k_j}{\left(-i\omega\right)}$ and $\eta = \frac{v_P\left(ik_z\right)}{\left(-i\omega\right)}$. Approximating (20) has

$$\eta = \left(1 + \xi^2\right)^{1/2} \approx \left(1 + \frac{\xi^2}{2}\right) \approx \frac{1 + a_2\xi^2}{1 + b_2\xi^2}. \tag{21}$$

After rearranging the above equation and equating like powers in ξ^2 (ignoring any powers of ξ^2 greater than or equal to ξ^4) it may be determined that $a_2 = 3/4$, so that $b_2 = 1/4$ and (20) may be written in the form

$$\frac{v_p\left(ik_z\right)}{\left(-i\omega\right)}=\left(1+\frac{v_P^2k_j^2}{\left(-i\omega\right)^2}\right)^{1/2}\approx\frac{1+\dfrac{3}{4}\dfrac{v_P^2k_j^2}{\left(-i\omega\right)^2}}{1+\dfrac{1}{4}\dfrac{v_P^2k_j^2}{\left(-i\omega\right)^2}}. \tag{22}$$

or

$$\frac{v_p\left(ik_z\right)}{\left(-i\omega\right)}\left(1+\frac{1}{4}\frac{v_P^2k_j^2}{\left(-i\omega\right)^2}\right)\approx 1+\frac{3}{4}\frac{v_P^2k_j^2}{\left(-i\omega\right)^2}. \tag{23}$$

After further rearrangement

$$\left(ik_z\right)\left(-i\omega\right)^2-\frac{\left(-i\omega\right)^3}{v_p}+\frac{v_P^2k_j^2\left(ik_z\right)}{4}-\frac{3v_pk_j^2\left(-i\omega\right)}{4}=0. \tag{24}$$

Introducing $\left(ik_z\right)\rightarrow\partial/\partial z$ and $\left(-i\omega\right)\rightarrow\partial/\partial t$ with equation (24) operating on ψ produces

$$\frac{\partial^3\psi}{\partial z\partial^2 t}-\frac{1}{v_p}\frac{\partial^3\psi}{\partial^3 t}+\frac{v_P^2k_j^2}{4}\frac{\partial\psi}{\partial z}-\frac{3v_pk_j^2}{4}\frac{\partial\psi}{\partial t}=0. \tag{25}$$

This is known as the 45 degree approximation which may be put in a useable finite code using equations (16) – (19). From numerical modeling tests it has been found that except for exceptional cases, the 15 degree approximation is sufficient for the damping of spurious reflections from the model bottom.

NUMERICAL RESULTS

A simple 4 layer acoustic model over a halfspace is used to check the usefulness of the absorbing boundary condition for the model bottom. The model is described in Table 1 and in the two panels in Figure (1). The first test is a vertical seismic profile recording geometry. The receivers are set at depths of $0.25km$ to $1.75km$ at $10m\left(0.01km\right)$ intervals. The source is located at the surface in the borehole (zero offset). The upper panel in Figure (2) is the synthetic seismogram with no damping at the boundary bottom while the lower panel has damping introduced. Using the same model, except that the source and receivers are located at the surface, synthetics are computed. The receivers are located at offsets of $0.0km$ to $1.75km$ at $25m\left(0.025km\right)$ intervals. The upper panel of Figure (3) has no damping while the bottom panel has damping introduce. Numerical experiments found that the 15 degree paraxial approximation to the wave equation produced reasonable results. Consequently, although derived and programmed, the 45 degree approximation has not been implemented to this point.

CONCLUSIONS

An absorbing boundary condition at the model bottom was derived for the case of an acoustic wave. Radial symmetry was assumed for the equation of motion and a finite Hankel transform was applied to remove the dependence on the radial coordinate. A finite difference problem in depth (z) and time (t) remained. In a manner similar to that presented in Clayton and Engquist (1977) both the 15 degree and 45 degree absorbing condition were obtained for this transformed problem at the model bottom removing spurious reflections from that artificial boundary. Numerical experiments showed that the 15 degree approximation was sufficient for most problems. Examples of the implementation of this method were shown for two types of recording; vertical seismic profile (VSP) acquisition geometry and the case where the source and receivers were located on the surface.

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ACKNOWLEDGEMENTS

The first author wishes to thank the sponsors of CREWES and NSERC (Professor G.F. Margrave, CRDPJ 379744-08) for financial support in undertaking this work.

Layer	Thickness	V_P	Density
1	0.4	1.8750	2.00
2	0.4	2.2020	2.25
3	0.4	2.5000	2.34
4	0.4	3.3060	2.44
Hspace	---	4.0000	2.50

Table 1: The parameters of the model used in the included figures. Thicknesses are in *km*, velocities in *km/s* and density in *gm/cm³.*

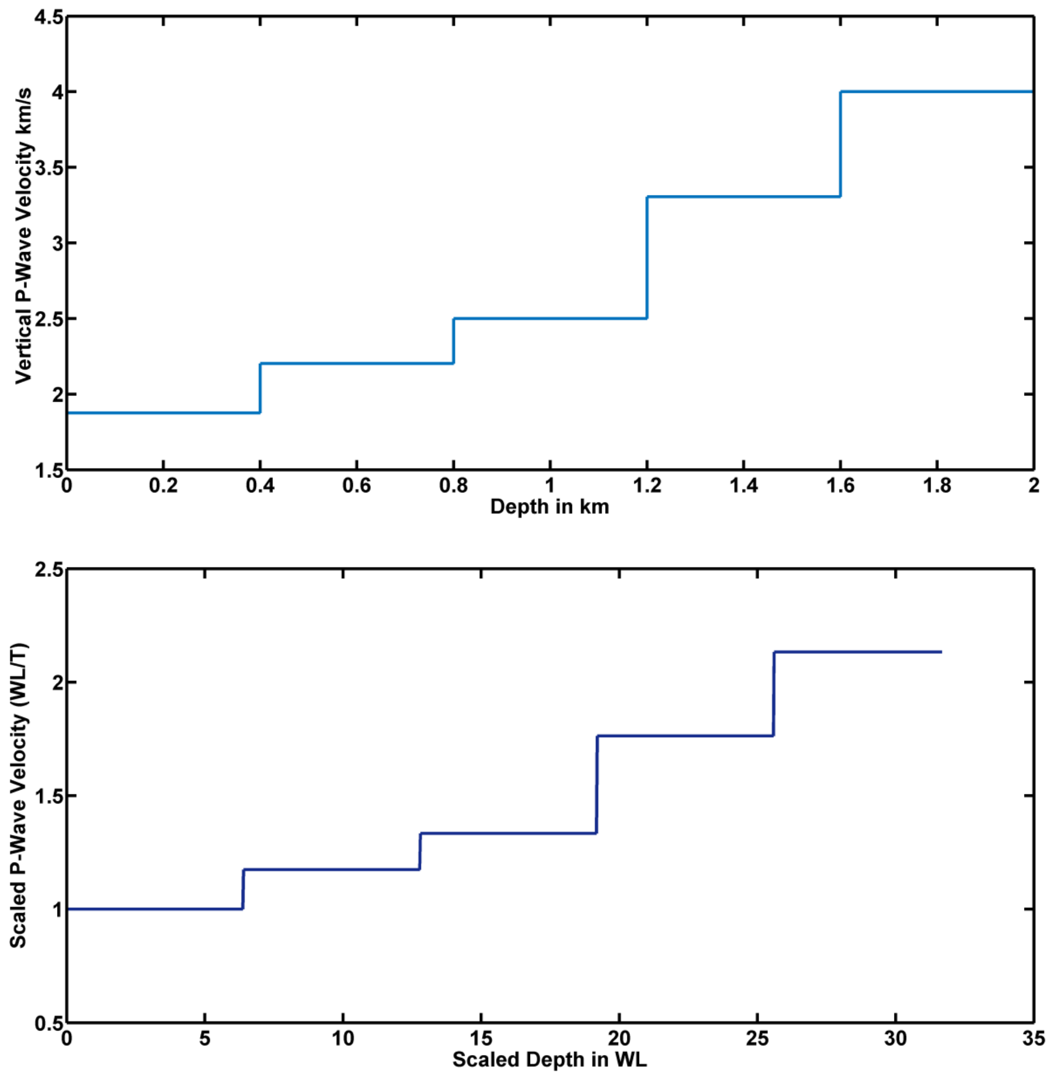

Figure 1: The vertical *P* – wave velocity in *km/s* plotted versus depth in *km* in the upper panel. In the lower panel the scaled *P* – wave velocity velocity is plotted versus the number of depth grid points.

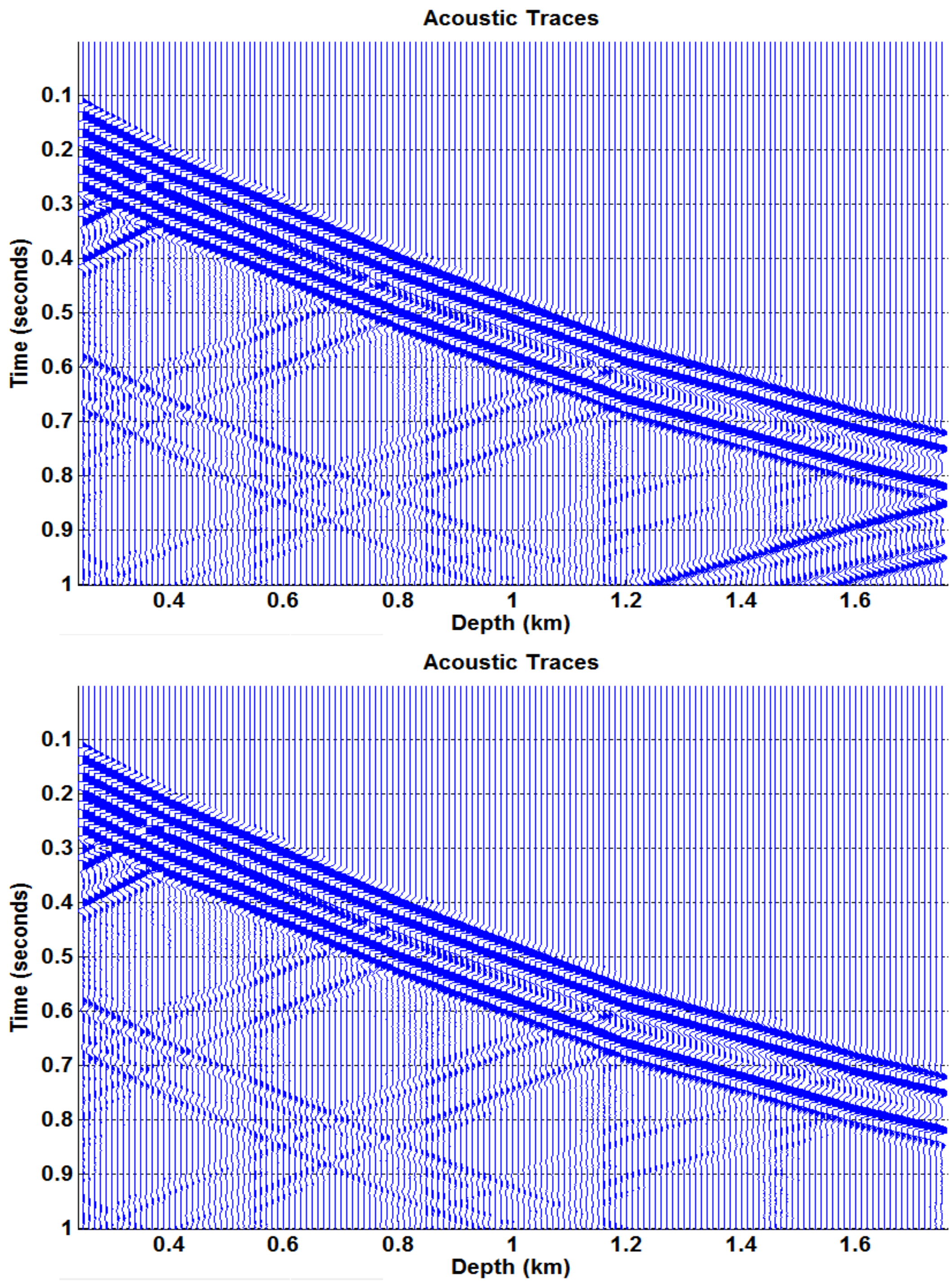

Figure 2: VSP synthetic traces for both source and receivers in the borehole. The source is at a depth of 75*m* (0.075*km*). The upper panel shows the synthetic with no absorbing conditions and the lower panel has an absorbing boundary condition at the model bottom.

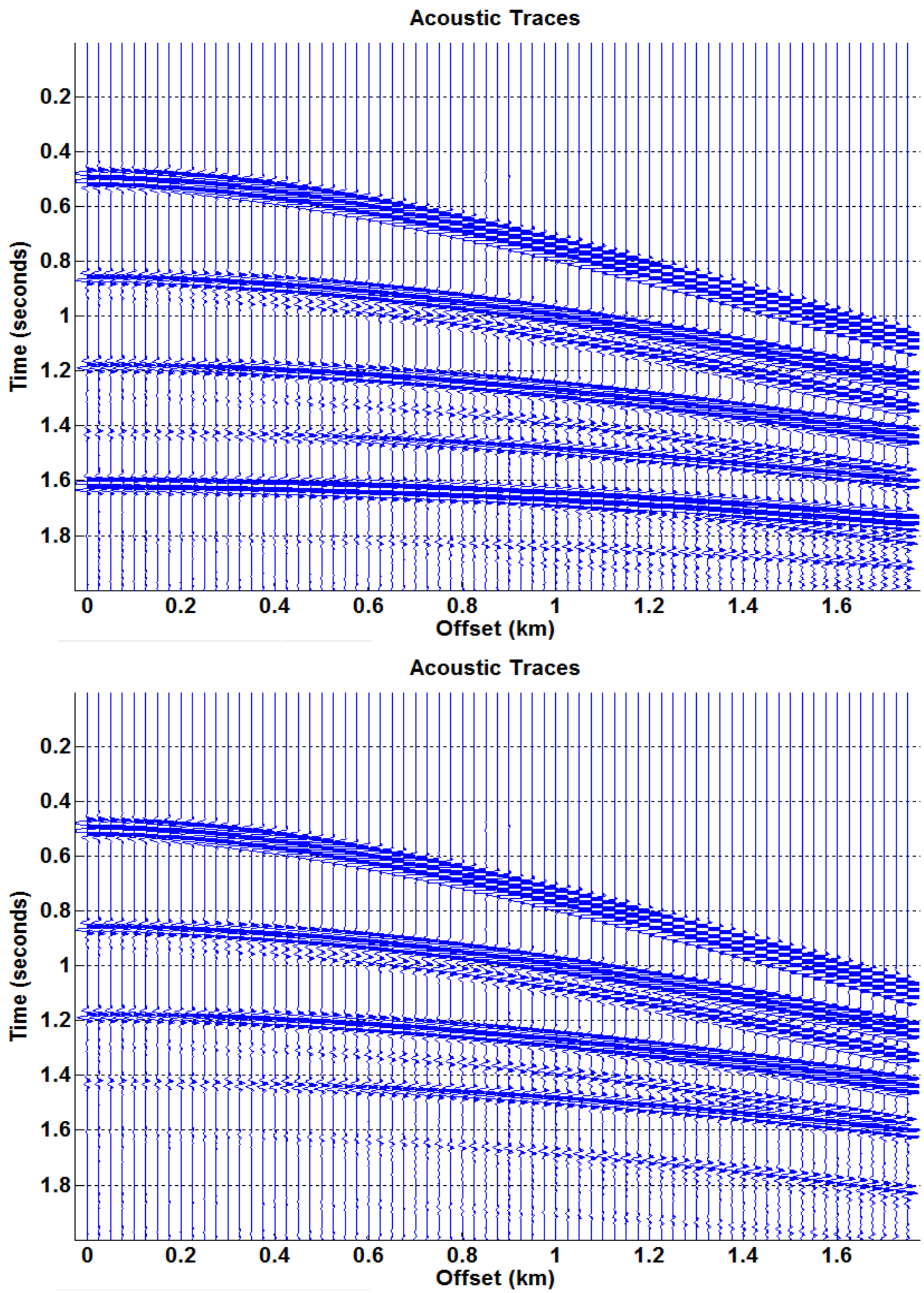

Figure 3: Offset synthetic traces for both source and receivers at the surface are shown. The upper panel shows synthetic traceswith no absorbing conditions and the lower panel has an absorbing boundary condition at the model bottom.