Higher order approximate expressions for qP, qS_1 and qS_2 phase velocities in a weakly anisotropic orthorhombic medium

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ABSTRACT

Higher order linearized approximations of the phase velocities for the quasi – compressional (qP) and two quasi – shear wave types, qS_1 and qS_2, in a *weakly anisotropic* orthorhombic medium are presented. Some manipulation of the formulae obtained by standard linearization techniques is done so that the phase velocities are in the form consisting of the most degenerate cases phase velocities (ellipsoids) of an orthorhombic medium plus correction terms to compensate for the deviation from the degenerate orthorhombic case. This is analogous to the ellipsoidal case in a transversely isotropic medium. The quantities in the formulae for the phase velocities all have physical interpretations, that is, they can all be associated with some physically realizable quantity. Further, obtaining the related approximations for group velocities of the three wave propagation types is considerably simplified.

INTRODUCTION

The problem of obtaining linearized approximations for the qP and qS_1 and qS_2 phase velocities, or equivalently the eigenvalues, in an orthorhombic medium is addressed here. The formula presented by Backus (1965) is initially used for this purpose. Once this approximation is obtained a rearrangement of terms is done to put the anisotropic coefficients from the original formula for the qP and qS_1 and qS_2 phase velocities into alternate configurations. This is done so that each of the terms, or individual collection of terms in the expressions for the phase velocities, has a physical meaning or can be associated with some geometrical formalism. This facilitates undertakings such as determining methods to pursue for inversion of phase velocity data to obtain the anisotropic parameters that define the medium or to use the derived phase velocity approximations to obtain approximations for the group velocities.

PRELIMINARY THEORY

The square of the linearized quasi-compressional (qP) phase velocity, $v_{qP}(n_k)$, in a 21 parameter anisotropic medium may be written as (Backus, 1965), where the a_{ijkl} are the density normalized stiffness coefficients (c_{ijkl}/ρ) and have the dimensions of velocity squared and $\mathbf{n}$ is defined below

$$v_{qP}^2(\mathbf{n}) = a_{ijk\ell} n_i n_j n_k n_\ell \quad (i, j, k, \ell = 1,2,3). \tag{1}$$

Einstein summation is assumed and the quantities n_i are the components of the unit phase (wavefront normal) vector in the qP wavefront propagation direction defined in Cartesian coordinates as

$$\mathbf{n}=\left(n_1,n_2,n_3\right)=\left(\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta\right) \quad (2)$$

where is the polar angle θ is measured from the positive x_3 (vertical) axis and ϕ being the azimuthal angle measured in a positive sense from the x_1 axis.

In Voigt notation, after some minor derivations, (1) becomes, for an orthorhombic medium

$$v_{qP}^2\left(n_k\right)=A_{11}n_1^2+A_{22}n_2^2+A_{33}n_3^2+E_{12}n_1^2n_2^2+E_{13}n_1^2n_3^2+E_{23}n_2^2n_3^2 \quad (3)$$

The E_{ij} are the linearized anellipsoidal terms, specifying the deviation of the slowness, phase or ray (group) surfaces from the ellipsoidal, are defined as

$$E_{12}=2\left(A_{12}+2A_{66}\right)-\left(A_{11}+A_{22}\right). \quad (3)$$

$$E_{13}=2\left(A_{13}+2A_{55}\right)-\left(A_{11}+A_{33}\right). \quad (4)$$

$$E_{23}=2\left(A_{23}+2A_{44}\right)-\left(A_{22}+A_{33}\right). \quad (5)$$

These expressions could be compared to those given for a *mildly* anisotropic orthorhombic medium presented in Gassmann (1964) or Schoenberg and Helbig (1997) as an indication of the how linearization simplifies the phase velocity expression.

The square of the phase velocity $v^2\left(\mathbf{n}\right)$ of an arbitrary wave in an anisotropic medium is given by the expression (see, e.g., Červený, 2001):

$$v_{qP}^2\left(\mathbf{n}\right)=a_{ijk\ell}n_in_jg_kg_\ell\,. \quad (6)$$

The exact expression for the components V_i of the ray (group) velocity $\mathbf{V}$ reads (see again Červený, 2001):

$$V_i\left(\mathbf{n}\right)=c^{-1}a_{ijk\ell}n_\ell g_jg_k\,. \quad (7)$$

The quantity $v^2\left(\mathbf{n}\right)$ itself represents the eigenvalue of the Christoffel matrix corresponding to the wave being considered. The quantities g_i are the components of the corresponding eigenvector **g**. This eigenvector represents the polarization vector of the wave.

EQUATION OF MOTION

The substitution of an asymptotic solution

$$u_i\left(x_j,t\right)=\sum_{n=0}^{\infty}\frac{A_i\left(x_j\right)}{\left(i\omega\right)^n}\exp\left(-i\omega\left(t-\tau\left(x_j\right)\right)\right) \quad (6)$$

into the equation of motion in a general anisotropic media

$$\frac{\partial}{\partial x_j}\left(c_{ijkl}\frac{\partial u_k}{\partial x_l}\right)-\rho\frac{\partial^2 u_i}{\partial t^2}=0 \qquad (7)$$

results in, among other equations (Červený, 2001), the eikonal equation

$$\left(\Gamma_{jk}\left(x_j,n_i\right)-G_m\delta_{jk}\right)\mathbf{g}_m=0 \quad \left(m=1,2,3\right) \qquad (8)$$

This is a cubic equation in G, the three values for which, $G_m=c_m^2\left(n_i\right) \quad \left(m=1,2,3\right)$, are the eikonals, related to the phase velocities of three mode types of wave propagation, qP, qS_1 and qS_2, with $\mathbf{n}=\left(n_1,n_2,n_3\right)$, $\mathbf{n}=\left(\sin\theta\cos\phi,\sin\theta\sin\phi,\cos\theta\right)$. The vector $\mathbf{g}_m=\left(g_1^{(m)},g_2^{(m)},g_3^{(m)}\right)$ is the polarization (eigen) vector corresponding to the m^th^ propagation mode for the equations given in (8). Employing the definitions

$$p_j=\frac{\partial\tau\left(x_i\right)}{\partial p_j} \qquad (9)$$

$$\mathbf{p}=\left(p_1,p_2,p_3\right) \quad \text{and} \quad \mathbf{p}\cdot\mathbf{n}=c_m^{-1}\left(n_i\right) \quad \left(m=1,2,3\right) \qquad (10)$$

equation (8) could have been written as

$$\left(\Gamma_{jk}\left(x_j,p_i\right)-G_m\delta_{jk}\right)\mathbf{g}_m=0 \quad \left(G_m=1 \quad \left(m=1,2,3\right)\right) \qquad (11)$$

However, in what follows, $\Gamma_{jk}\left(\mathbf{n}\right)$ will be used and will not appear in the parameter list, as it is implied. $\mathbf{\Gamma}\left(\mathbf{n}\right)$ is the Christoffel matrix, a 3×3 symmetric matrix. Its components, Γ_{jk}, are given generally by

$$\Gamma_{jk}=a_{ijkl}n_in_l \qquad (6)$$

or in other equivalent forms if the symmetries of the subscripts of a_{ijkl} are used and specifically for an orthorhombic medium, in Voigt notation, as (Schoenberg and Helbig, 1996)

$$\Gamma_{11}=A_{11}n_1^2+A_{66}n_2^2+A_{55}n_3^2 \qquad (6)$$

$$\Gamma_{22}=A_{66}n_1^2+A_{22}n_2^2+A_{44}n_3^2 \qquad (7)$$

$$\Gamma_{33}=A_{55}n_1^2+A_{44}n_2^2+A_{33}n_3^2 \qquad (8)$$

$$\Gamma_{23}=\Gamma_{32}=\left(A_{23}+A_{44}\right)n_2n_3 \qquad (9)$$

$$\Gamma_{13} = \Gamma_{31} = \left(A_{13} + A_{55}\right) n_1 n_3 \tag{10}$$

$$\Gamma_{12} = \Gamma_{21} = \left(A_{12} + A_{66}\right) n_1 n_2 \tag{11}$$

This leads to the eikonal equations for the three values of G_m $\left(m = 1,2,3\right)$ corresponding to the three modes of wave propagation in an anisotropic medium and their related eigenvectors (polarization vectors) $\mathbf{g}_m$ $\left(m = 1,2,3\right)$

$$\left(\mathbf{\Gamma}\left(\mathbf{n}\right) - G_m \mathbf{I}\right)\mathbf{g}_m = 0 \tag{12}$$

The $3D$ qP phase velocity propagation direction vector, $\mathbf{n}$, has been previously defined.

A sequence of two orthonormal rotation transformations is applied to the Christoffel matrix $\mathbf{\Gamma}$. This transformation is given by

$$B_{rs} = \Gamma_{pq} a_{rp} a_{sq} \tag{21}$$

for some matrices a_{rp} and a_{qs}. This transformation can be shown to be equal to a double multiplication (inner product) of $\mathbf{\Gamma}$ by the two vectors, $\mathbf{e}^{(1)}$ and $\mathbf{e}^{(2)}$ (Every and Sachse, 1992, Pšenčík and Farra, 2005) as

$$B_{rs} = \Gamma_{pq} e_p^{(r)} e_q^{(s)} \tag{28}$$

with summation over repeated indices implied in the previous two equations. These vectors are required to be orthonormal to the to the qP phase velocity vector, $\mathbf{n}$, defined by

$$\mathbf{n} = \left(n_1, n_2, n_3\right) = \left(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta\right) \tag{14}$$

The vectors $\mathbf{e}^{(1)}$ and $\mathbf{e}^{(2)}$ are given as

$$\mathbf{e}^{(1)} = \left(e_1^{(1)}, e_2^{(1)}, e_3^{(1)}\right) = \left(\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta\right) \tag{15}$$

$$\mathbf{e}^{(2)} = \left(e_1^{(2)}, e_2^{(2)}, e_3^{(2)}\right) = \left(-\sin\phi, \cos\phi, 0\right) \tag{16}$$

The choice of the vectors $\mathbf{e}^{(1)}$ and $\mathbf{e}^{(2)}$ is not arbitrary as mentioned above as there is the following condition on the orthonormal vector triad (Jech and Pšenčík, 1989)

$$\left(\mathbf{e}^{(1)}, \mathbf{e}^{(2)}, \mathbf{n}\right) \equiv \left(\mathbf{e}^{(1)}, \mathbf{e}^{(2)}, \mathbf{e}^{(3)}\right) \tag{17}$$

That is, the vectors $\mathbf{e}^{(1)}$ and $\mathbf{e}^{(2)}$, which are orthonormal to each other, must lie in a plane to which $\mathbf{n} = \mathbf{e}^{(3)}$ is normal. As it is easily shown that

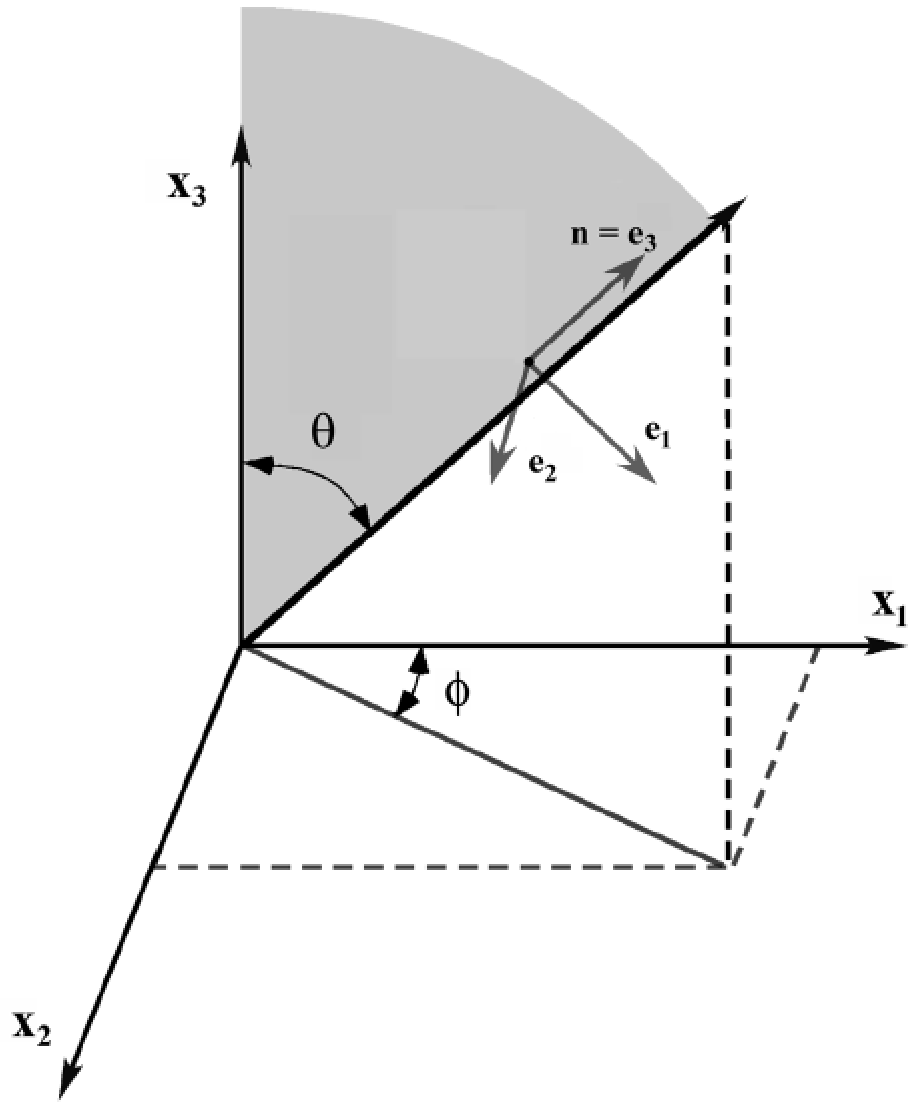

Fig. 1. Orthonormal triad of vectors $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 = \mathbf{n})$. The choice of the orientation of the orthonormal vector pair $(\mathbf{e}_1, \mathbf{e}_2)$ which spans a plane normal to $\mathbf{e}_3 = \mathbf{n}$ is arbitrary. However, for the problem being considered here, $\mathbf{e}_1$ has been chosen to be oriented in such a manner that it and $\mathbf{e}_3 = \mathbf{n}$ form the plane of ray propagation for a transversely isotropic medium. This degenerate arrangement allows the angle ϕ to be arbitrary. Consequently, it is chosen equal to zero so that $\mathbf{e}_2$ is normal to the $(\mathbf{e}_1, \mathbf{e}_3)$ plane and can be taken to describe the direction of particle displacement of the qS_H wave.

$$\sin\phi = n_2/D \quad \text{and} \quad \cos\phi = n_1/D, \tag{18}$$

$\mathbf{e}^{(1)}$ and $\mathbf{e}^{(2)}$ may be written in the following manner

$$\mathbf{e}^{(1)} = \left(n_1 n_3, n_2 n_3, n_3^2 - 1\right) / D \tag{19}$$

$$\mathbf{e}^{(2)} = \left(-n_2, n_1, 0\right) / D \tag{20}$$

with

$$D = \left(n_1^2 + n_2^2\right)^{1/2} \tag{21}$$

and

$$n_1^2 + n_2^2 + n_3^2 = 1 \tag{22}$$

When numerically implementing the above equations, the possibility of $D \equiv 0$ exists. This does not mean that the term in question, with D in the denominator, is numerically equal to ∞.

The matrix $\mathbf{B}$ is also symmetric. The determination of it elements is tedious. The final results are given in Appendix A. What hasn't changed from these rotations is the new equation

$$\mathbf{B}(\mathbf{n}) - G_m \mathbf{I} = 0 \qquad \left[\left(\mathbf{B}(\mathbf{n}) - G_m \mathbf{I}\right)\mathbf{g}_m = 0\right] \tag{23}$$

still produces the eigenvalues corresponding to the phase velocities of the three modes of propagation in an (anisotropic) orthorhombic medium. In the above it is assumed that $\mathbf{g}_m$ are not identically equal to zero. It is helpful to write the matrix $\mathbf{B}$

$$\mathbf{B}(\mathbf{n}) = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{12} & B_{22} & B_{23} \\ B_{13} & B_{23} & B_{33} \end{bmatrix} \tag{24}$$

The first approximation to the qP eigenvalue is (equation (3))

$$\lambda_{qP} = v_{qP}^2\left(n_k\right) = B_{33} = A_{11}n_1^2 + A_{22}n_2^2 + A_{33}n_3^2 + E_{12}n_1^2 n_2^2 + E_{13}n_1^2 n_3^2 + E_{23}n_2^2 n_3^2 \tag{25}$$

which together with the assumption, based on numerical experimentation indicating that the off – diagonal terms of $\mathbf{B}$ are significantly smaller than the diagonal terms, has

$$\left(B_{11} - G\right)\left(B_{22} - G\right) - B_{12}^2 = 0 \tag{26}$$

used in Pšenčík and Farra (2005) among earlier papers in the literature leading to

$$G^2 - \left(B_{11} + B_{22}\right)G + B_{11}B_{22} - B_{12}^2 = 0 \tag{27}$$

and subsequently to

$$G=\left\{\left(B_{11}+B_{22}\right)\pm\left[\left(B_{11}+B_{22}\right)^{2}-4\left(B_{11}B_{22}-B_{12}^{2}\right)\right]^{1/2}\right\}\Big/2$$
$$G_{qS_1}\Big/G_{qS_2}=\left\{\left(B_{11}+B_{22}\right)\pm\left[\left(B_{11}-B_{22}\right)^{2}+4B_{12}^{2}\right]^{1/2}\right\}\Big/2 \qquad (28)$$

As with the qP case, the first approximations to the qS phase velocities are obtained as

$$v_{qS_1}^2\Big/v_{qS_2}^2=\left\{\left(B_{11}+B_{22}\right)\pm\left[\left(B_{11}-B_{22}\right)^{2}+4B_{12}^{2}\right]\right\}\Big/2 \qquad (29)$$

The sum and subsequent mean, $\left(v_{qS}^2\right)^M$ of the two qS phase velocities obtained from equation (29) is similar in form to equation (25) for the qP phase velocity, have the forms

$$2\left(v_{qS}^2\right)^M=B_{11}+B_{22}=\left(A_{55}+A_{66}\right)n_1^2+\left(A_{44}+A_{66}\right)n_2^2+\left(A_{44}+A_{55}\right)n_3^2-E_{12}n_1^2n_2^2-E_{13}n_1^2n_3^2-E_{23}n_2^2n_3^2 \qquad (30)$$

$$v_{qS}^M=\left\{\frac{1}{2}\left[\left(A_{55}+A_{66}\right)n_1^2+\left(A_{44}+A_{66}\right)n_2^2+\left(A_{44}+A_{55}\right)n_3^2-E_{12}n_1^2n_2^2-E_{13}n_1^2n_3^2-E_{23}n_2^2n_3^2\right]\right\}^{1/2} \qquad (31)$$

It is not difficult to see that equation (31) is very similar to (3). As a consequence a reasonable approximation to the mean shear wave velocity maybe determined in a manner similar to that employed in Daley and Krebes (2006) to obtain an approximation to the qP group velocity, $V_{qP}\left(\mathbf{N}\right)$, in an orthorhombic medium. In earlier works, such as Song and Every (2000), similar results were obtained by *intuitive methods*.

$$\frac{1}{V_{qP}^2\left(N_k\right)}=\frac{N_1^2}{A_{11}}+\frac{N_2^2}{A_{22}}+\frac{N_3^2}{A_{33}}-\frac{E_{12}N_1^2N_2^2}{A_{11}A_{22}}-\frac{E_{13}N_1^2N_3^2}{A_{11}A_{33}}-\frac{E_{23}N_2^2N_3^2}{A_{22}A_{33}} \qquad (32)$$

where $\mathbf{N}$ is the propagation direction of the ray (group velocity) with Θ being the polar measured from the positive x_3 $\left(\text{vertical}\right)$ axis and Φ the azimuthal angle measured in a positive from the x_1 axis.

$$\mathbf{N}=\left(N_1,N_2,N_3\right)=\left(\sin\Theta\cos\Phi,\sin\Theta\sin\Phi,\cos\Theta\right). \qquad (33)$$

Thus, in a similar manner the squared mean of the group velocity of the two shear wave modes may be approximated as

$$\frac{1}{\left(V_{qS}^2\right)^M}=\frac{1}{2}\left[\frac{N_1^2}{\left(A_{55}+A_{66}\right)}+\frac{N_2^2}{\left(A_{44}+A_{66}\right)}+\frac{N_3^2}{\left(A_{44}+A_{55}\right)}+\frac{E_{12}N_1^2N_2^2}{A_{11}A_{22}}+\frac{E_{13}N_1^2N_3^2}{A_{11}A_{33}}+\frac{E_{23}N_2^2N_3^2}{A_{22}A_{33}}\right] \tag{34}$$

HIGHER ORDER APPROXIMATION

The Jacobi method of determining the eigenvalues of a symmetric matrix involves a number of orthonormal rotations of the matrix $\mathbf{B}$ so that it becomes numerically close to a diagonal matrix, which essentially solves the problem. However, as the direction of $\mathbf{n}$ is desired not to change, only one rotation will be used. As it was previously mentioned that $\mathbf{e}^{(1)}$ and $\mathbf{e}^{(2)}$ are orthonormal, the Given's rotation (Press et al., 1994) will be in that plane (normal to $\mathbf{n}$), at an angle ξ, transforming $\mathbf{B}$ to $\hat{\mathbf{B}}$, chosen such that $\hat{B}_{12}=0$ under the presumption that $B_{11}>B_{22}$. Let $\mathbf{R}$ be this matrix given by

$$\mathbf{R}=\begin{bmatrix}\cos\xi & \sin\xi & 0\\ -\sin\xi & \cos\xi & 0\\ 0 & 0 & 1\end{bmatrix} \tag{35}$$

and define the new matrix, after rotation, to be

$$\hat{\mathbf{B}}=\mathbf{RBR}^{-1} \tag{36}$$

The first element of $\hat{\mathbf{B}}$ to be calculated is $\hat{B}_{12}$ as it was chosen as the consequence of a degree of freedom to be zero. Thus

$$\hat{B}_{12}=\hat{B}_{21}=\left(B_{22}-B_{11}\right)\sin\xi\cos\xi+B_{12}\left(\cos^2\xi-\sin^2\xi\right)=0 \tag{37}$$

and the angle ξ is given by

$$\tan 2\xi=\frac{2B_{12}}{\left(B_{11}-B_{22}\right)}\quad\left[\operatorname{ctn}2\xi=\frac{\left(B_{11}-B_{22}\right)}{2B_{12}}\right] \tag{38}$$

The remaining terms of the matrix $\hat{\mathbf{B}}$ are given by

$$\begin{aligned}\hat{B}_{11}&=B_{22}\sin^2\xi+B_{11}\cos^2\xi+2B_{12}\sin\xi\cos\xi\\&=B_{22}\sin^2\xi+B_{11}\cos^2\xi\end{aligned} \tag{38}$$

$$\begin{aligned}\hat{B}_{22}&=B_{11}\sin^2\xi+B_{22}\cos^2\xi-2B_{12}\sin\xi\cos\xi\\&=B_{11}\sin^2\xi+B_{22}\cos^2\xi\end{aligned} \tag{40}$$

$$\hat{B}_{13} = \hat{B}_{31} = B_{13}\cos\xi + B_{23}\sin\xi \tag{41}$$

$$\hat{B}_{23} = \hat{B}_{32} = -B_{13}\sin\xi + B_{23}\cos\xi \tag{42}$$

$$\hat{B}_{33} = B_{33} \tag{43}$$

The eigenvalue problem in $\hat{\mathbf{B}}$ now has the form

$$\begin{bmatrix} \hat{B}_{11} & 0 & \hat{B}_{13} \\ 0 & \hat{B}_{22} & \hat{B}_{23} \\ \hat{B}_{13} & \hat{B}_{23} & \hat{B}_{33} \end{bmatrix} - G\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 0\,. \tag{44}$$

Expanding equation (44) yields

$$\left(\hat{B}_{11}-G\right)\left(\hat{B}_{22}-G\right)\left(\hat{B}_{33}-G\right)-\hat{B}_{13}^{2}\left(\hat{B}_{22}-G\right)-\hat{B}_{23}^{2}\left(\hat{B}_{11}-G\right)=0 \tag{45}$$

As the first eigenvalue of interest is the one dealing with qP wave propagation, rewrite (45) as

$$\left(\hat{B}_{33}-G\right)-\frac{\hat{B}_{13}^{2}}{\left(\hat{B}_{11}-G\right)}-\frac{\hat{B}_{23}^{2}}{\left(\hat{B}_{22}-G\right)}=0 \tag{46}$$

The initial approximation to G in the previous equation was $G=\hat{B}_{33}=B_{33}$. This value is used in the second and third terms. Denote G in the first term as $\bar{G}$ to indicate a higher order approximation results in

$$\bar{G}_{qP}=\hat{B}_{33}-\frac{\hat{B}_{13}^{2}}{\left(\hat{B}_{11}-\hat{B}_{33}\right)}-\frac{\hat{B}_{23}^{2}}{\left(\hat{B}_{22}-\hat{B}_{33}\right)} \tag{47}$$

With this done, the updated value of the qP phase velocity in an orthorhombic medium is

$$\bar{v}_{qP}^{2}\left(\mathbf{n}\right)=\hat{B}_{33}+\frac{\hat{B}_{13}^{2}}{\left(\hat{B}_{33}-\hat{B}_{11}\right)}+\frac{\hat{B}_{23}^{2}}{\left(\hat{B}_{33}-\hat{B}_{22}\right)} \tag{48}$$

where all quantities in (48) have been defined.

What should next be considered is the two modes of shear waves qS_1 and qS_2. As in the qP case, start with equation (45) and alter it to read

$$\left(\hat{B}_{11}-G\right)-\frac{\hat{B}_{13}^{2}}{\left(\hat{B}_{33}-G\right)}-\frac{\hat{B}_{23}^{2}\left(\hat{B}_{11}-G\right)}{\left(\hat{B}_{22}-G\right)\left(\hat{B}_{33}-G\right)}=0\,. \tag{49}$$

Here, the previous value of G is taken to be $\hat{B}_{11}$ and again the updated value of G is that value with an over score so that

$$\overline{G}_{qS_1} = \hat{B}_{11} - \frac{\hat{B}_{13}^2}{\left(\hat{B}_{33} - \hat{B}_{11}\right)} \tag{50}$$

and the new value of the phase velocity for this shear wave mode is given as

$$\overline{v}_{qS_1}^2\left(\mathbf{n}\right) = \hat{B}_{11} - \frac{\hat{B}_{13}^2}{\left(\hat{B}_{33} - \hat{B}_{11}\right)} \tag{51}$$

In a like manner, the updated eigenvalue for the second shear wave mode is

$$\overline{G}_{qS_2} = \hat{B}_{22} - \frac{\hat{B}_{23}^2}{\left(\hat{B}_{33} - \hat{B}_{22}\right)} \tag{52}$$

with the updated phase velocity being

$$\overline{v}_{qS_2}^2\left(\mathbf{n}\right) = \hat{B}_{22} - \frac{\hat{B}_{23}^2}{\left(\hat{B}_{33} - \hat{B}_{22}\right)} \tag{53}$$

CONCLUSIONS

The eigenvalue/eigenvector problem posed by the Christoffel equation is considered in an effort to obtain reasonable approximations for the compressional shear wave phase velocities in a weakly anisotropic (orthorhombic) medium. A fairly straightforward method was proposed and produced results which have a similar form to those obtained by others by more complex and rigorous means (for example: Pšenčík and Gajewski, 1998, Pšenčík and Farra, 2005, Pšenčík and Farra, 2016). The expressions obtained may be employed for a variety of uses in seismic data processing. What has not been dealt with here is determination of the corresponding eigenvectors. It was thought that this might lead to some marginal confusion. Interested persons may used a variety of available software to numerically determine the polarization (eigen) vectors.

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APPENDIX A

$$B_{11} = A_{44}\frac{n_2^2}{\left(n_1^2+n_2^2\right)} + A_{55}\frac{n_1^2}{\left(n_1^2+n_2^2\right)} + E_{12}\frac{n_1^2 n_2^2 n_3^2}{\left(n_1^2+n_2^2\right)} - E_{13}n_1^2 n_3^2 - E_{23}n_2^2 n_3^2 \quad \text{(A.1)}$$

$$B_{22} = A_{66}\left(n_1^2+n_2^2\right) + \frac{A_{55}n_3^2 n_2^2}{\left(n_1^2+n_2^2\right)} + \frac{A_{44}n_3^2 n_1^2}{\left(n_1^2+n_2^2\right)} - \frac{E_{12}n_1^2 n_2^2}{\left(n_1^2+n_2^2\right)} \quad \text{(A.2)}$$

From the above two equations it should be noted that

$$\left[B_{11} + B_{22} = A_{44}\left(n_2^2+n_3^2\right) + A_{55}\left(n_1^2+n_3^2\right) + A_{66}\left(n_1^2+n_2^2\right) - E_{12}n_1^2 n_2^2 - E_{13}n_1^2 n_3^2 - E_{23}n_2^2 n_3^2\right] \quad \text{(A.3)}$$

$$\left[B_{11} + B_{22} = \left(A_{55}+A_{66}\right)n_1^2 + \left(A_{44}+A_{66}\right)n_2^2 + \left(A_{44}+A_{55}\right)n_3^2 - E_{12}n_1^2 n_2^2 - E_{13}n_1^2 n_3^2 - E_{23}n_2^2 n_3^2\right] \quad \text{(A.4)}$$

Continuing

$$B_{33} = A_{11}n_1^2 + A_{22}n_2^2 + A_{33}n_3^2 + E_{12}n_1^2 n_2^2 + E_{13}n_1^2 n_3^2 + E_{23}n_2^2 n_3^2 \quad \text{(A.5)}$$

$$B_{12} = A_{22}n_2^2 n_3 - n_1 n_2 n_3\left[A_{66} - \frac{\left(A_{44}-A_{55}\right)n_3^2}{\left(n_1^2+n_2^2\right)} - \frac{E_{12}n_1^2}{\left(n_1^2+n_2^2\right)}\right] \quad \text{(A.6)}$$

$$B_{13} = n_3\left[\left(A_{11}-A_{55}\right)n_1^2 + \left(A_{22}-A_{44}\right)n_2^2 + E_{12}n_1^2 n_2^2 + E_{13}n_1^2 n_3^2 + E_{23}n_2^2 n_3^2\right]\Big/D \quad \text{(A.7)}$$

$$B_{23} = n_1 n_2\left[-A_{11} + A_{22}n_2^2 + A_{66}\left(n_1^2+n_2^2\right) + A_{55}n_3^3 - A_{44}n_3^3 - E_{12}n_2^2 - E_{13}n_3^2 + E_{23}n_3^2\right]\Big/D \quad \text{(A.8)}$$