The relationship between scattering theory and the reflection coefficient for elastic media: Forward scattering series

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ABSTRACT

The scattering theory can be used as a powerful theoretic approach to understand and process seismic data. Exploring inverse scattering series, which have been used to remove multiples from seismic data, depends on understanding how these series generate primaries and multiples. The inverse scattering methods depend on an understanding of how the forward scattering series generates primaries and multiples. In this work, we study the forward scattering series for elastic media in order to identify on which terms in inverse scattering series are important for imaging and inversion. Primary reflections are described by all of the terms in the series excluding the first term.

INTRODUCTION

The inverse scattering series is a direct multi-D inversion method that can perform the tasks associated with multiple removal, imaging and inversion (Weglein et al., 2005; Weglein, 2006a) The inverse scattering method has a direct response for imaging and inversion problems for a large contrast and a multi-D corrugated target. The advantages of this method are involves explicit algorithms which directly provide improved estimates for medium properties without recourse to highly non-linear optimization procedures and determines data requirements for non-linear direct parameter estimation. In this paper we study the forward scattering series for elastic media in order to identify on which terms in the inverse scattering series are important for performing imaging and inversion. Forward scattering series for one parameter and two parameter acoustic media are considered as analytically. We will extend the acoustic results by consider the forward scattering series for elastic media and we show how the forward scattering series create a solution for elastic wave equation.

THE LIPPMANN-SCHWINGER EQUATION

The forward scattering series is designed to characterize the wavefield produced by a localized source and propagated through an Earth model. The forward scattering series is solved using a boundary value approach by adding an infinite number of terms that corresponding to propagations in the reference medium. These separate solutions separated by different orders of scattering interactions with a point scatter Earth model.

The Lippmann-Schwinger equation is (Matson, 1997; Weglein et al., 2003)

$$P = G_0 + G_0 V P \tag{1}$$

where P is the wavefield in the actual medium, and G_0 is the reference medium green's function, and V represents the difference in properties between the reference and true medium. The Lippmann-Schwinger equation can be written as a series expansion.

$$
\begin{aligned}
P &= G_0 + G_0 V P \\
&= G_0 + G_0 V (G_0 + G_0 V P) \\
&= G_0 + G_0 V G_0 + G_0 V G_0 V P \\
&= (I + G_0 V + G_0 V G_0 V + \cdots) G_0 \\
&= P_0 + P_1 + P_2 + \cdots
\end{aligned} \tag{2}
$$

The first term in the Born series is the reference Green's function, represents a direct wave propagating in the reference medium from source at x_s, to the measurement point at x_g. The higher order terms in the Born series contains $V(x)$ between wavefiels propagating in the reference medium for difference number of scattering interaction.

SCATTERING POTENTIAL FOR ELASTIC MEDIA (SINGLE INTERFACE)

The elastic scattering potential is defined as (Weglein and Stolt, 2012):

$$
\mathcal{V}_E = L - L_0 = \begin{pmatrix} \mathcal{V}_{pp} & \mathcal{V}_{psI} & \mathcal{V}_{psII} \\ \mathcal{V}_{sIp} & \mathcal{V}_{sIsI} & \mathcal{V}_{sIsII} \\ \mathcal{V}_{sIIp} & \mathcal{V}_{sIIsI} & \mathcal{V}_{sIIsII} \end{pmatrix} \tag{3}
$$

where SI=SH and SII=SV.

The elements of this matrix are

$$
\mathcal{V}_{ii} = \rho_0 \big(\omega^2 a_\rho + \alpha_0^2 \partial_i a_\gamma \partial_i + \beta_0^2 \textstyle\sum_{i \neq j} \partial_j a_\mu \, \partial_j \big) \qquad i, j = x, x, z \tag{4}
$$

$$
\mathcal{V}_{ij} = \rho_0 \big(\alpha_0^2 \partial_i a_\gamma \partial_j - 2\beta_0^2 \partial_i a_\mu \partial_j + \beta_0^2 \partial_j a_\mu \partial_i \big) \qquad j \neq i \tag{5}
$$

The SH to SV and SV to SH scattering potentials are zero

$$
\mathcal{V}_{SHSV} = \mathcal{V}_{SVSH} = 0 \tag{6}
$$

Also the SH to P and P to SH scattering potentials are zero

$$
\mathcal{V}_{SHP} = \mathcal{V}_{PSH} = 0 \tag{7}
$$

Hence, the elastic scattering potential can be written as

$$\mathcal{V}_E = L - L_0 = \begin{pmatrix} \mathcal{V}_{pp} & 0 & \mathcal{V}_{psII} \\ 0 & \mathcal{V}_{sIsI} & 0 \\ \mathcal{V}_{sIIp} & 0 & \mathcal{V}_{sIIsII} \end{pmatrix} \tag{8}$$

FORWARD SCATTERING SERIES

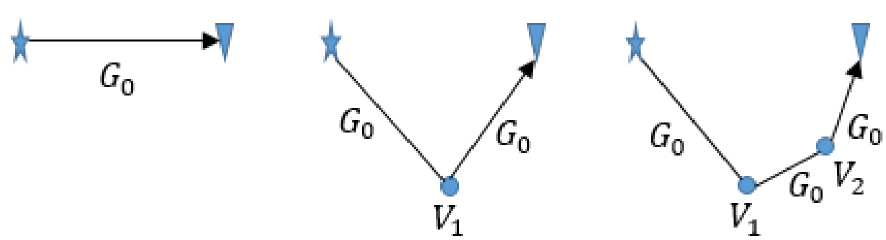
In exploration seismology, a forward problem is designed to characterize the wavefield emanating from a source and propagating through an earth model. In this model, the earth includes of layers with constant velocities and discontinues velocities at boundaries. The forward problem for this layered model is solved using a boundary value approach in which the solutions are constant velocities for each layer. These separate solutions are matched at the boundary on each consequent layer.

We consider the two dimensional Born series (Innanen, 2009):

$$\begin{aligned} P(x_g, z_g, x_s, z_s, \omega) &= G_0(x_g, z_g, x_s, z_s, \omega) \\ &+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G_0(x_g, z_g, x', z', \omega) V(x', z') G_0(x', z', x_s, z_s, \omega) dx' dz' \\ &+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G_0(x_g, z_g, x', z', \omega) V(x', z') \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G_0(x', z', x'', z'', \omega) \\ &\times V(x', z') G_0(x'', z'', x_s, z_s, \omega) dx'' dz'' + \cdots \\ &= P_0 + P_1 + P_2 + \cdots \end{aligned} \tag{9}$$

This equation plays a pivotal role in scattering theory. Based on this equation the wavefield in an actual medium is the sum of the wavefield in a reference medium and integral that represent the scattered wavefield due to perturbation.

The Born series can be shown as:



THE REFLECTION CASE

The P-to P scattering by a single interface

The pp scattering potential in terms of velocity and density perturbations is

$$\mathcal{V}_{PP} = -\rho_0\omega^2[a_\alpha + a_\rho\left(1+\cos\sigma - \frac{2\beta_0^2}{\alpha_0^2}sin^2\sigma\right) - a_\beta\frac{2\beta_0^2}{\alpha_0^2}sin^2\sigma] \tag{10}$$

The first order term in the Born series is given by (when $z_g < z_1$)

$$P_1(x_g, z_g, x_s, z_s, \omega) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G_0(x_g, z_g, x', z', \omega)V(z')G_0(x', z', x_s, z_s, \omega)dx'dz' \tag{11}$$

where x_g, z_g and x_s, z_s are respectively the position of the receiver and source. The function G_0 describes propagation in the reference medium, and can be written as a 2D Green's function bilinear form.

$$G_0(x_g, z_g, x', z', \omega) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}dk_x'\int_{-\infty}^{\infty}dk_z'\frac{e^{ik_x'(x_g-x')}e^{ik_z'(z_s-z')}}{k^2-k_x'^2-k_z'^2} \tag{12}$$

$$G_0(x', z', x_s, z_s, \omega) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}dk_x''\int_{-\infty}^{\infty}dk_z''\frac{e^{ik_x''(x'-x_s)}e^{ik_z''(z'-z_s)}}{k^2-k_x''^2-k_z''^2} \tag{13}$$

After the Fourier transform over x_g and x_s on both side of Eq.11, we have

$$\hat{P}1(k_g, z_g, -k_s, z_s, \omega) =$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\hat{G}_0(k_g, z_g, x', z', \omega)V(z')\hat{G}_0(x', z', -k_s, z_s, \omega)dk_x'dk_z'$$

$$= -\frac{\rho_0\omega^2}{4q_gq_s}(\frac{1}{2\pi})^2e^{-i(q_gz_g+q_sz_s)}\int_{-\infty}^{\infty}dk_x'\int_{-\infty}^{\infty}dk_z''e^{-i(k_g-k_s)x'}Ve^{i(q_g+q_s)z'} \tag{14}$$

where $q_g^2 = k^2 - k_g^2$ and $q_s^2 = k^2 - k_s^2$. Eq.14 also can be written as

$$\hat{P}1\big(k_g, z_g, -k_s, z_s, \omega\big) =$$
$$= -\frac{C_P k^2}{4q_g^2}\left(\frac{1}{2\pi}\right)^2 e^{-iq_g(z_g+z_s)}\int_{-\infty}^{\infty} dk'_x \int_{-\infty}^{\infty} dk''_z V e^{i2q_g z'}$$
$$= -\frac{C_P}{4cos^2\theta}\left(\frac{1}{2\pi}\right)^2 e^{-iq_g(z_g+z_s)}\int_{-\infty}^{\infty} dk'_x \int_{-\infty}^{\infty} dk''_z V e^{i2q_g z'} \tag{16}$$

Finally , $\hat{P}1$ can be obtain as

$$\hat{P}1\big(k_g, z_g, -k_s, z_s, \omega\big) =$$
$$= -\frac{C_P}{4cos^2\theta} e^{-iq_g(z_g+z_s)}[a_\alpha(-2q_g) + a_\rho(-2q_g)\left(1 + \cos\sigma - \frac{2\beta_0^2}{\alpha_0^2} sin^2\sigma\right) -$$
$$a_\beta(-2q_g)\frac{2\beta_0^2}{\alpha_0^2} sin^2\sigma] \tag{17}$$

The higher order terms in the Born series have an important role when the perturbation value is lager while the higher order terms become less important for small value of perturbation and tha Born approximation is valid.

$$\hat{P}_{Born} \cong \hat{P}_0 + \hat{P}_1 \tag{18}$$

In this expression, the first term propagates outward from the source directly to receiver. The second term is a reflected wave. The reflection coefficient, which is the ration of the amplitude of the incident and reflected wave, for the Born approximation can be written as ($z_g = z_s = 0$)

$$R_1^{PP}(\theta) \approx \frac{1}{2(1+cos\sigma)} a_\alpha + \left(\frac{1}{2} - \frac{\beta_0^2}{\alpha_0^2}(1 - cos\sigma)\right) a_\rho - \frac{\beta_0^2}{\alpha_0^2}(1 - cos\sigma) a_\beta \tag{19}$$

where $\sigma = 2\theta$ is the opening angle.

NUMERICAL EXAMPLE: SINGLE INTERFACE

The result of approximation is compared to the exact equation. The geologic parameters are given in Table 1. The reflection coefficient is calculated and plotted against the opening angle up to the critical angle. The approximation is compared to the exact Zoeppritz equation for small layer contrast (model 1 and 2) and large layer contrast (model 3 and 4). Figure 1 shows the comparison of the approximation for small layer contrast (model 1). We can see that the approximation is in a good agreement with the exact equation up to angle range from 65°. After this angle, there is deviations near the critical angle. The exact curve is complex beyond the critical angle, while the linearized curve is real and decreasing for all opening angles before the critical angle. Figure 2 shows the comparison of the approximation for model 2. There is a small deviation from the exact solution, while model 2 is a small layer contrast.

For large layer contrasts the deviations become larger as compared to small layer contrast models (Figs. 3 and 4). In this paper, we investigated the first order term in the Born series to obtain the reflection coefficient. By adding the upper order of Born series the deviation from exact equation can be decrease for large layer contrast.

Table 1. The geologic parameters for four models.

Model	$V_{P0}(m/s)$	$V_{S0}(m/s)$	$\rho_0(kg/m^3)$	$V_{P1}(m/s)$	$V_{S1}(m/s)$	$\rho_1(kg/m^3)$
1	3000	1800	2200	3100	1900	2250
2	3000	1800	2200	3200	2000	2250
3	3000	1800	2200	4000	2500	2400
3	3000	1800	2200	4500	3000	2600

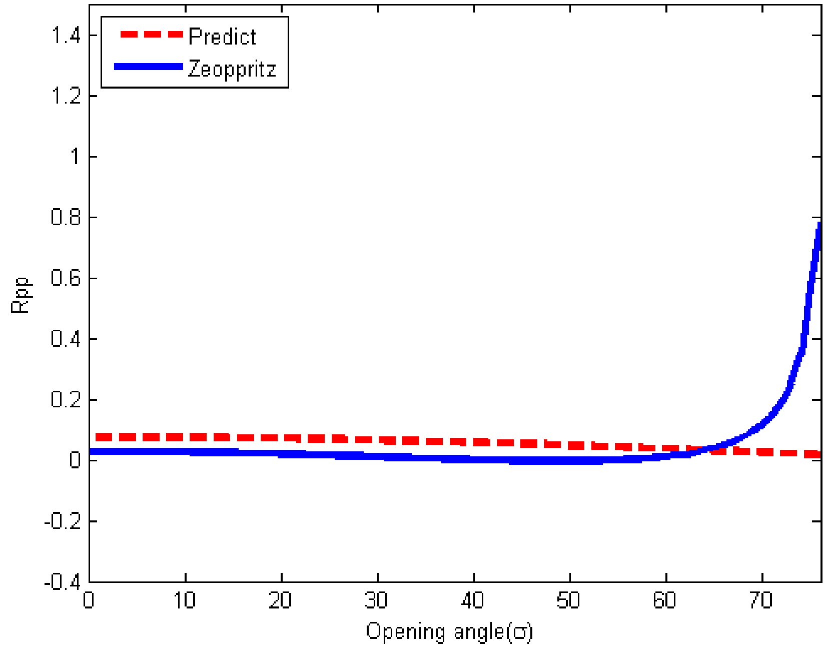


Figure 1: The comparison between the synthesized values and the actual values of Rpp for a sample model 1.

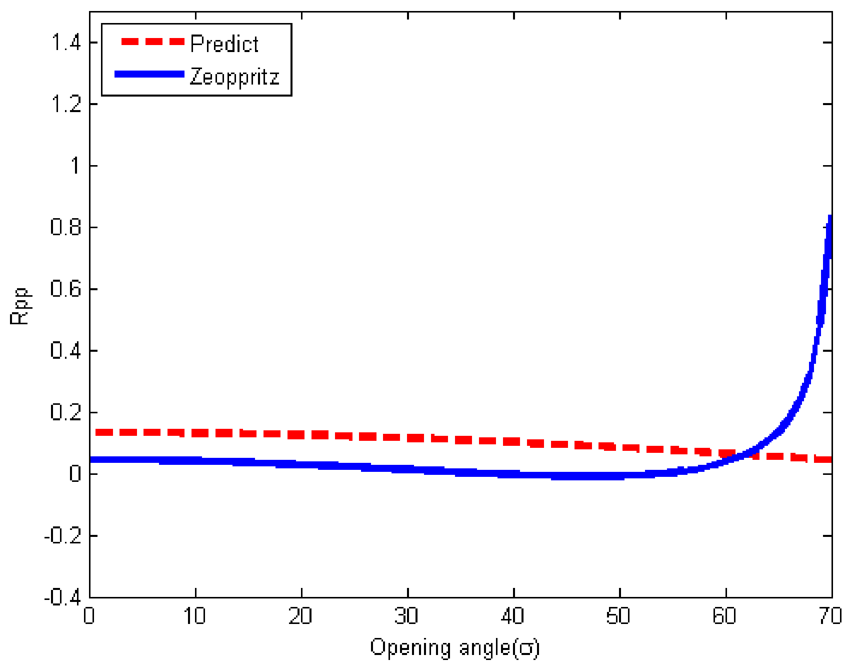


Figure 2: The comparison between the synthesized values and the actual values of Rpp for a sample model 2.

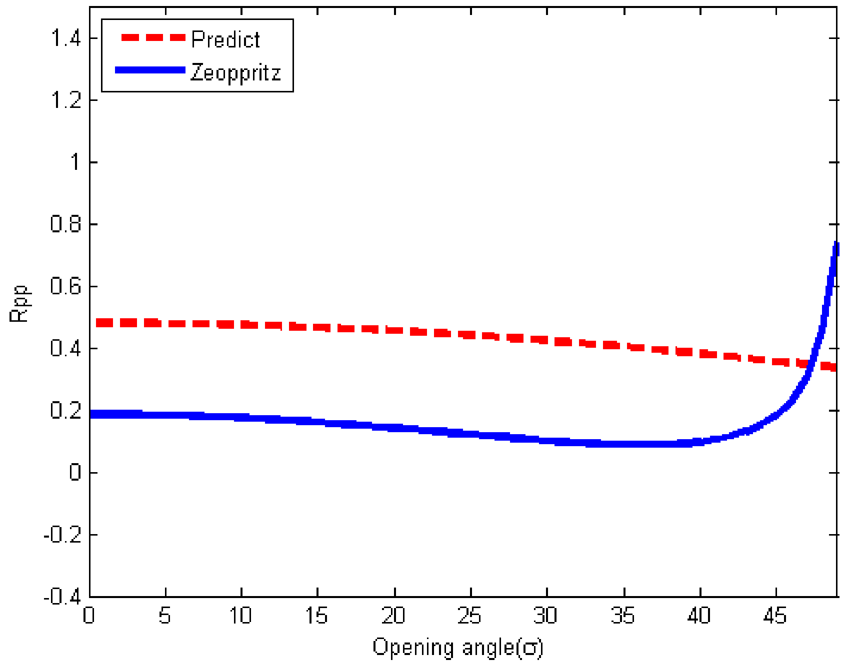


Figure 3: The comparison between the synthesized values and the actual values of Rpp for a sample model 3.

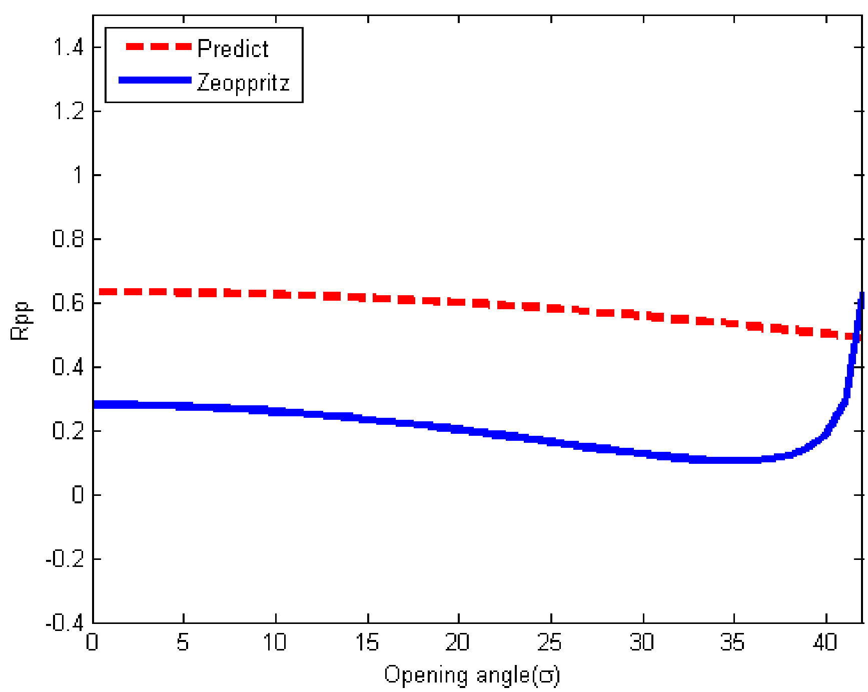


Figure 4: The comparison between the synthesized values and the actual values of Rpp for a sample model 4.

CONCLUSIONS

The scattering theory is applied to investigate a mapping method between the earth model and seismic data. The Born series is established and full series terms are derived. Theses series were able to predict and interpret seismic reflection data including primary and multiple events. To identify on which terms in inverse scattering series are important for imaging and inversion the forward scattering series for elastic media is investigated. The results show that the exact curve of R_{PP} is complex beyond critical angles, while the approximation curve reminds real and decreasing for all opening angle that is smaller than the critical angles.

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