Elastic wave equations in modified patchy-saturated media and its numerical simulation

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ABSTRACT

Based on the modified patchy-saturated porous model, according to the method Biot used for the foundation of elastic wave equations in porous media, we established the stress-strain relations and obtained the dissipation function and kinetic energy in patchy-saturated porous media. By applying the Lagrange's equations, we derived the elastic wave equations in modified patchy-saturated media. Through changing the equations to first-order stress-velocity equations, we deduced the 3D high-order finite difference schemes and numerically solved the equations in the complex domain. Numerical results show that there are two kinds of P-waves and one S-wave in patchy saturated media. The energy of the slow P-wave is very weak in the "solid" phase of the patchy saturated porous media and can hardly be seen, even though it is stronger in the fluid phase. The fast P-wave and S-wave are clear both in the solid phase and the fluid phase. The slow P-wave has a high dispersion. The velocities of the three waves are consistent with the theoretical results. All these results show that our numerical method is correct and effective.

INTRODUCTION

As we all know, seismic attenuation and velocity dispersion are important properties which may be used both in exploration geophysics and development geophysics. Therefore, many scientists have studied seismic attenuation and velocity dispersion based on different models (e.g., Biot, 1956a, 1956b, 1962; Dutta and Odé, 1979a, 1979b; Mavko and Nur, 1979; Lopatnikov and Gurevich, 1988; Berryman, 1988; Chapman et al., 2006; Müller et al., 2010; Gurevich and Makarynska, 2012; Kuteynikova et al., 2014; Qi et al., 2014; Tisato and Quintal, 2014; Yao et al., 2015; Zhang and He, 2015; Spencer and Shine, 2016). The main basic models are the Biot model (Biot, 1956a, 1956b), the BISQ model (Dvorkin and Nur, 1993; Dvorkin et al., 1994) and the patchy-saturated model (White, 1975). The Biot model is the earliest model that considers the relative motion between solid and fluid. It is the relative motion between matrix and fluid that causes the presence of the slow P-wave. Due to its high attenuation and dispersion, it is difficult to observe in reality. However, scientists do verify its existence both in the laboratory (Plona, 1980; Mou, 1996) and through numerical modeling (Wang 1990; Zhu and McMechan, 1991; Zeng et al., 2001; Arntsen and Carcione, 2001; Zhang et al., 2004; Müller et al., 2011). Unfortunately, the attenuation expected from the Biot model is much smaller than

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that measured in real sandstones (Grant, 1994). In order to better describe the wave propagation in fluid-containing rocks, the squirt mechanism (Mavko and Nur, 1979) is also considered besides Biot flow, and the BISQ model was established (Dvorkin and Nur, 1993). However, attenuation computed from the BISQ model is still underestimated within the seismic band (Yang et al., 2007; Nie et al., 2012; Ling et al., 2013; Li et al., 2013). Since White (1975) proposed a model for seismic wave attenuation which was shown to cause high attenuation coefficients in the exploration frequency band(10-150Hz), more and more scientists began to study seismic wave attenuations based on the White model (e.g., Dutta, N. C., and A. J. Seriff, 1979; Dutta and Odé, 1979a, 1979b; Carcione et al., 2003). In fact, the White model is a patchy-saturated model, where rock pores are saturated with different types of fluids and mesoscopic flow can occur. Pride et al. (2004) showed that the squirt flow mechanism does not adequately describe attenuation in the seismic frequency range, whereas the mesoscale flow model can account for the attenuation in the low frequency range. Mesoscopic flow can occur within a wide range of scales in typical sedimentary rocks, from the largest pore size to the smallest wavelength, and therefore can cause attenuation over a broad range of frequencies (Müller et al., 2010). Hence, the White model is an ideal representation of a patchy-saturated model with mesoscopic fluid flow.

Based on the White model, we (Zhang and He, 2015) established a patchy saturated model, called the modified patchy-saturated model, and derived its P-wave equations on the basis of Biot's equations and Johnson's bulk modulus. We derived the seismic wave attenuation coefficients by solving the equations, and analyzed the characteristics of wave attenuation at low frequencies. The results show that there is a high attenuation and velocity dispersion in modified patchy-saturated porous media within the seismic band. This indicates that the modified patchy-saturated model may better describe seismic waves in real rocks. In order to fully understand seismic waves in modified patchy-saturated model, it is necessary to establish elastic wave equations in which the S-wave is also included.

In this paper, we will establish elastic wave equations in a modified patchy-saturated model following the method of Biot's equations established in 1962 (Biot, 1962). As numerical simulation is a useful way to study wave propagation in fluid-containing rocks, especially in complicated media, such as patchy-saturated media, we will numerically solve the equations by using a high-order finite difference method and simulate the wave propagations in modified patchy-saturated media.

ESTABLISHMENT OF ELASTIC WAVE EQUATIONS IN MODIFIED PATCHY-SATURATED MODEL

The modified patchy-saturated model is illustrated in Figure 1(Zhang and He, 2015). In this model, some regions were fully saturated with one fluid and others were fully saturated with another fluid.

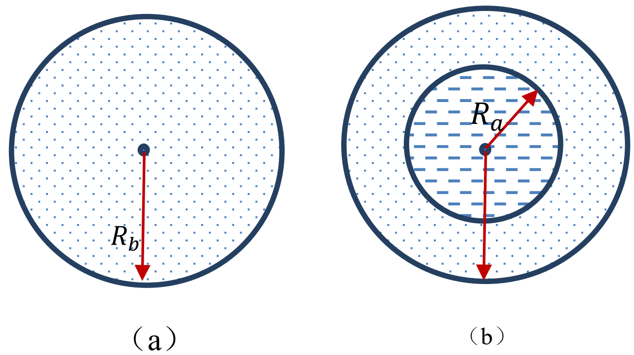

FIG.1. Modified patchy-saturated model for porous media

Figure 1a shows the rock skeleton and two kinds of fluids where the dots represent for fluid 1 and dashes represent for fluid 2. For convenient calculation, we consider Figure 1a as Figure 1b, which consists of two concentric spheres with radii R_a and R_b. In the smaller sphere, there are pores saturated with fluid 2(gas pocket in White model). In the larger sphere, there is the rock skeleton and the pores filled with fluid 1. In this model, we assumed that: The seismic wavelength is much larger than the gas pocket size and there is no interaction between two gas pockets (which is also White's assumption), and there is no movement between fluid 1 and the skeleton, but there is relative movement between fluid 2 and the skeleton.

First, we discuss the stress-strain relations in a modified patchy-saturated medium.

Biot analyzed the strain energy of a porous elastic medium and established the corresponding stress-strain relations (Biot, 1962). By using the same method, we can obtain the stress-strain relations in a modified patchy-saturated medium, as follows:

$$\sigma_{xx}=2\mu(\partial u_x/\partial x)+\lambda\theta+2\gamma D\varepsilon, \tag{1}$$

$$\sigma_{yy}=2\mu(\partial u_y/\partial y)+\lambda\theta+2\gamma D\varepsilon, \tag{2}$$

$$\sigma_{zz}=2\mu(\partial u_z/\partial z)+\lambda\theta+2\gamma D\varepsilon, \tag{3}$$

$$\sigma_{yz}=\mu[(\partial u_y / \partial z)+(\partial u_z / \partial y)], \tag{4}$$

$$\sigma_{xz}=\mu[(\partial u_x / \partial z)+(\partial u_z / \partial x)], \tag{5}$$

$$\sigma_{xy}=\mu[(\partial u_x / \partial y)+(\partial u_y / \partial x)], \tag{6}$$

$$s=2\gamma D\theta+2D\varepsilon. \tag{7}$$

If we let

$$e_{ij}=\begin{bmatrix} \frac{\partial u_x}{\partial x} & \frac{1}{2}(\frac{\partial u_x}{\partial y}+\frac{\partial u_y}{\partial x}) & \frac{1}{2}(\frac{\partial u_x}{\partial z}+\frac{\partial u_z}{\partial x}) \\ \frac{1}{2}(\frac{\partial u_x}{\partial y}+\frac{\partial u_y}{\partial x}) & \frac{\partial u_y}{\partial y} & \frac{1}{2}(\frac{\partial u_y}{\partial z}+\frac{\partial u_z}{\partial y}) \\ \frac{1}{2}(\frac{\partial u_x}{\partial z}+\frac{\partial u_z}{\partial x}) & \frac{1}{2}(\frac{\partial u_y}{\partial z}+\frac{\partial u_z}{\partial y}) & \frac{\partial u_z}{\partial z} \end{bmatrix}, \tag{8}$$

then, equations (1) - (6) can be written using the following abbreviated notation

$$\sigma_{ij}=2\mu e_{ij}+\delta_{ij}(\lambda\theta+2\gamma D\varepsilon), \tag{9}$$

where, u_x, u_y, u_z are the displacement components of the "solid"(frame rock containing fluid 1) of the patchy-saturated porous rock; θ is the volume strain of the "solid", and $\theta=\nabla\cdot\vec{u}$ with $\vec{u}=(u_x,u_y,u_z)$; ε is the volume strain of fluid 2 relative to "solid", and $\varepsilon=\nabla\cdot\vec{w}$ with $\vec{w}=\phi(\vec{U}-\vec{u})$, where $\vec{U}$ is the displacement of fluid 2, and ϕ is the porosity of the rock; $\sigma_{ij}(i,j=x,y,\mathrm{z})$ are the total stress components of the porous rock; s is the stress component of fluid 2; δ_{ij} is delta function; μ,λ,γ,D are elastic coefficients of the rock and they satisfy the following relations

$$H=\lambda+2\mu=K+\frac{4}{3}\mu, \tag{10}$$

$$\gamma=1-\frac{\overline{K}_m}{\overline{K}_s}, \tag{11}$$

$$D=\frac{\overline{K}_s}{2}\left[\gamma+\frac{S_2\phi}{K_{f_2}}(\overline{K}_s-K_{f_2})\right]^{-1}, \tag{12}$$

where H is the plane-wave modulus of partially-saturated rock, K and μ are bulk and shear moduli of the patchy-saturated rock, respectively; $\overline{K}_m$ and $\overline{K}_s$ are the bulk moduli of the skeleton containing fluid 1, but excluding fluid 2 and the frame filled with fluid 1, respectively; K_{f_2} and S_2 are the bulk modulus and saturation of fluid 2, respectively.

Then, we examine the dissipation function of a modified patchy-saturated medium.

If we define the rate of flow of fluid 2 by the time derivative of the volume flow vector:

$$\frac{\partial \overline{w}}{\partial t} = \left(\dot{w}_x, \dot{w}_y, \dot{w}_z\right), \tag{13}$$

then, the dissipation function in per unit volume of the patchy-saturated rock is

$$F_D = \frac{1}{2}\eta_2(r_{11}\dot{w}_x{}^2 + r_{22}\dot{w}_y{}^2 + r_{33}\dot{w}_z{}^2 + 2r_{23}\dot{w}_y\dot{w}_z + 2r_{31}\dot{w}_z\dot{w}_x + 2r_{12}\dot{w}_x\dot{w}_y), \tag{14}$$

where η_2 is the viscosity of fluid 2.

The symmetric matrix

$$\left[r_{ij}\right] = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{12} & r_{22} & r_{23} \\ r_{13} & r_{23} & r_{33} \end{bmatrix} \tag{15}$$

represents a flow resistivity, whereas its inverse

$$\left[r_{ij}\right]^{-1} = \left[\kappa_{ij}\right] = \begin{bmatrix} \kappa_{11} & \kappa_{12} & \kappa_{13} \\ \kappa_{12} & \kappa_{22} & \kappa_{23} \\ \kappa_{13} & \kappa_{23} & \kappa_{33} \end{bmatrix}, \tag{16}$$

also symmetric, represents a "permeability matrix." (Biot, 1962)

When the medium is isotropic,

$$\begin{gathered} \kappa_{11} = \kappa_{22} = \kappa_{33} = \kappa \\ \kappa_{12} = \kappa_{13} = \kappa_{23} = 0, \end{gathered} \tag{17}$$

and Equation (14) becomes

$$F_D = \frac{1}{2}\frac{\eta_2}{\kappa}(\dot{w}_x{}^2 + \dot{w}_y{}^2 + \dot{w}_z{}^2), \tag{18}$$

where the quantity κ is the permeability of the medium.

Next, we discuss the kinetic energy of a unit volume of the porous medium.

At low frequencies, Biot gave the vector $\bar{\upsilon}$ which determines the components of the relative microvelocity field in the pores:

$$\begin{aligned}\upsilon_x &= a_{11}\dot{w}_x + a_{12}\dot{w}_y + a_{13}\dot{w}_z \\ \upsilon_y &= a_{21}\dot{w}_x + a_{22}\dot{w}_y + a_{23}\dot{w}_z \\ \upsilon_z &= a_{31}\dot{w}_x + a_{32}\dot{w}_y + a_{33}\dot{w}_z,\end{aligned} \tag{19}$$

where the coefficients a_{ij} depend on the coordinates in the pores and the pore geometry.

The kinetic energy of a unit volume of the patchy-saturated medium is given by

$$T = \frac{1}{2}\rho_1(\dot{u}_x{}^2 + \dot{u}_y{}^2 + \dot{u}_z{}^2) + \frac{1}{2}\rho_{f_2}\iiint_\Omega[(\dot{u}_x + \upsilon_x)^2 + (\dot{u}_y + \upsilon_y)^2 + (\dot{u}_z + \upsilon_z)^2]d\Omega. \tag{20}$$

In this expression, ρ_1 represents the mass density of the “solid”(frame rock with fluid 1), and ρ_{f_2} represents the mass density of fluid 2 in patchy-saturated medium.

After some derivation, equation (20) can also be written as

$$T = \frac{1}{2}\rho(\dot{u}_x{}^2 + \dot{u}_y{}^2 + \dot{u}_z{}^2) + \rho_{f_2}(\dot{u}_x\dot{w}_x + \dot{u}_y\dot{w}_y + \dot{u}_z\dot{w}_z) + \frac{1}{2}\rho_{f_2}\iiint_\Omega(\upsilon_x{}^2 + \upsilon_y{}^2 + \upsilon_z{}^2)d\Omega, \tag{21}$$

where ρ is the total mass of the porous medium per unit volume.

From equation (19), we derive

$$\rho_{f_2}\iiint_\Omega(\upsilon_x{}^2 + \upsilon_y{}^2 + \upsilon_z{}^2)d\Omega = \sum_{ij} m_{ij}\dot{w}_i\dot{w}_j, \tag{22}$$

with

$$m_{ij} = \rho_{f_2}\iiint_\Omega(\sum_k a_{ki}a_{kj})d\Omega. \tag{23}$$

It is obvious that we have

$$m_{ij}=m_{ji}. \tag{24}$$

For a medium with statistical isotropy of the microvelocity field, the coefficients m_{ij} reduce to

$$m_{ij}=m\delta_{ij}. \tag{25}$$

Then, we get the kinetic energy of a unit volume

$$T=\frac{1}{2}\rho(\dot{u}_x{}^2+\dot{u}_y{}^2+\dot{u}_z{}^2)+\rho_{f_2}(\dot{u}_x\dot{w}_x+\dot{u}_y\dot{w}_y+\dot{u}_z\dot{w}_z)+\frac{1}{2}m(\dot{w}_x{}^2+\dot{w}_y{}^2+\dot{w}_z{}^2). \tag{26}$$

Finally, by applying Lagrange's equations, we can derive the elastic wave equations in modified patchy-saturated medium.

The Lagrange's equations are written as

$$\begin{aligned}\sum_j\frac{\partial\sigma_{ij}}{\partial x_j}&=\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{u}_i}\right)\\ \frac{\partial s}{\partial x_i}&=\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{w}_i}\right)+\frac{\partial F_D}{\partial\dot{w}_i}.\end{aligned} \tag{27}$$

These also are the general dynamical equations when gravity forces are neglected.

For the isotropic medium, substituting expressions (18) and (26) for F_D and T, and substituting the stress-strain relations (7) and (9) into equation (27), we obtain the elastic wave equations in a modified patchy-saturated medium

$$\begin{aligned}&2\sum_j\frac{\partial}{\partial x_j}(\mu e_{ij})+\frac{\partial}{\partial x_i}(\lambda\theta+2\gamma D\varepsilon)=\frac{\partial^2}{\partial t^2}(\rho u_i+\rho_{f_2}w_i)\\ &\frac{\partial}{\partial x_i}(2\gamma D\theta+2D\varepsilon)=\frac{\partial^2}{\partial t^2}(\rho_{f_2}u_i+mw_i)+\frac{\eta_2}{\kappa}\frac{\partial w_i}{\partial t}.\end{aligned} \tag{28}$$

In equation (28), the parameter ρ is the total mass of the modified patchy-saturated porous medium per unit volume, which satisfy the following

$$\rho=(1-\phi)\rho_s+\phi S_1\rho_{f_1}+\phi S_2\rho_{f_2}, \tag{29}$$

where ρ_s is the mass of the pure solid per unit volume, S_1 and ρ_{f_1} are saturation and mass density of fluid 1 in modified patchy-saturated medium, respectively.

For the elastic coefficients constant in an isotropic medium, the above equations can be written as

$$\begin{aligned}
&\mu\nabla^2\vec{u} + (H-\mu)\nabla\theta + 2\gamma D\nabla\varepsilon = \frac{\partial^2}{\partial t^2}(\rho\vec{u} + \rho_{f_2}\vec{w}) \\
&\nabla(2\gamma D\theta + 2D\varepsilon) = \frac{\partial^2}{\partial t^2}(\rho_{f_2}\vec{u} + m\vec{w}) + \frac{\eta_2}{\kappa}\frac{\partial\vec{w}}{\partial t}.
\end{aligned} \tag{30}$$

HIGH-ORDER FINITE DIFFERENCE SOLUTION OF ELASTIC WAVE EQUATIONS

Computational accuracy and efficiency are important factors of forward modeling for a wave field. The simulation of elastic waves involves more computation than P-waves, which leads to lower computational efficiency. In order to improve the efficiency of computation, we change equation (30) to the formula expressed by stress and velocity, i.e., first-order stress-velocity equations

$$\rho\frac{\partial v_x}{\partial t} + \rho_{f_2}\frac{\partial W_x}{\partial t} = \frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{xy}}{\partial y} + \frac{\partial\sigma_{xz}}{\partial z}, \tag{31}$$

$$\rho\frac{\partial v_y}{\partial t} + \rho_{f_2}\frac{\partial W_y}{\partial t} = \frac{\partial\sigma_{xy}}{\partial x} + \frac{\partial\sigma_{yy}}{\partial y} + \frac{\partial\sigma_{yz}}{\partial z}, \tag{32}$$

$$\rho\frac{\partial v_z}{\partial t} + \rho_{f_2}\frac{\partial W_z}{\partial t} = \frac{\partial\sigma_{xz}}{\partial x} + \frac{\partial\sigma_{yz}}{\partial y} + \frac{\partial\sigma_{zz}}{\partial z}, \tag{33}$$

$$\rho_{f_2}\frac{\partial v_x}{\partial t} + m\frac{\partial W_x}{\partial t} + \frac{\eta_2}{\kappa}W_x = \frac{\partial s}{\partial x}, \tag{34}$$

$$\rho_{f_2}\frac{\partial v_y}{\partial t} + m\frac{\partial W_y}{\partial t} + \frac{\eta_2}{\kappa}W_y = \frac{\partial s}{\partial y}, \tag{35}$$

$$\rho_{f_2}\frac{\partial v_z}{\partial t} + m\frac{\partial W_z}{\partial t} + \frac{\eta_2}{\kappa}W_z = \frac{\partial s}{\partial z}. \tag{36}$$

In the above equations, v_x, v_y, v_z are components of velocity of the "solid" $\vec{v}$ in x, y, z directions; $\vec{W} = \frac{\partial\vec{w}}{\partial t}$ and W_x, W_y, W_z are components of $\vec{W}$ in x, y, z directions, respectively.

Then, by applying the partial derivative to t in equations (1) - (7), we can change the form of the stress-strain relations to stress-velocity equations

$$\frac{\partial \sigma_{xx}}{\partial t} = H(\frac{\partial \nu_x}{\partial x} + \frac{\partial \nu_y}{\partial y} + \frac{\partial \nu_z}{\partial z}) - 2\mu(\frac{\partial \nu_y}{\partial y} + \frac{\partial \nu_z}{\partial z}) + 2\gamma D(\frac{\partial W_x}{\partial x} + \frac{\partial W_y}{\partial y} + \frac{\partial W_z}{\partial z}), \tag{37}$$

$$\frac{\partial \sigma_{yy}}{\partial t} = H(\frac{\partial \nu_x}{\partial x} + \frac{\partial \nu_y}{\partial y} + \frac{\partial \nu_z}{\partial z}) - 2\mu(\frac{\partial \nu_z}{\partial z} + \frac{\partial \nu_x}{\partial x}) + 2\gamma D(\frac{\partial W_x}{\partial x} + \frac{\partial W_y}{\partial y} + \frac{\partial W_z}{\partial z}), \tag{38}$$

$$\frac{\partial \sigma_{zz}}{\partial t} = H(\frac{\partial \nu_x}{\partial x} + \frac{\partial \nu_y}{\partial y} + \frac{\partial \nu_z}{\partial z}) - 2\mu(\frac{\partial \nu_x}{\partial x} + \frac{\partial \nu_y}{\partial y}) + 2\gamma D(\frac{\partial W_x}{\partial x} + \frac{\partial W_y}{\partial y} + \frac{\partial W_z}{\partial z}), \tag{39}$$

$$\frac{\partial \sigma_{\mathrm{yz}}}{\partial t} = \mu(\frac{\partial \nu_y}{\partial z} + \frac{\partial \nu_z}{\partial y}), \tag{40}$$

$$\frac{\partial \sigma_{\mathrm{xz}}}{\partial t} = \mu(\frac{\partial \nu_x}{\partial z} + \frac{\partial \nu_z}{\partial x}), \tag{41}$$

$$\frac{\partial \sigma_{\mathrm{xy}}}{\partial t} = \mu(\frac{\partial \nu_x}{\partial y} + \frac{\partial \nu_y}{\partial x}), \tag{42}$$

$$\frac{\partial s}{\partial t} = 2\gamma D(\frac{\partial \nu_x}{\partial x} + \frac{\partial \nu_y}{\partial y} + \frac{\partial \nu_z}{\partial z}) + 2\,\mathrm{D}(\frac{\partial W_x}{\partial x} + \frac{\partial W_y}{\partial y} + \frac{\partial W_z}{\partial z}). \tag{43}$$

We define V_x, V_y, V_z as the components of the velocity of fluid 2 $\vec{V}$ in three directions, then equations (31) - (36) can be written as

$$(\rho_{f_2}^2 - m\rho)\frac{\partial \nu_x}{\partial t} = \phi\rho_{f_2}\frac{\eta_2}{\kappa}\nu_x - \phi\rho_{f_2}\frac{\eta_2}{\kappa}V_x + \rho_{f_2}\frac{\partial s}{\partial x} - m(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z}), \tag{44}$$

$$(\rho_{f_2}^2 - m\rho)\frac{\partial \nu_y}{\partial t} = \phi\rho_{f_2}\frac{\eta_2}{\kappa}\nu_y - \phi\rho_{f_2}\frac{\eta_2}{\kappa}V_y + \rho_{f_2}\frac{\partial s}{\partial y} - m(\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z}), \tag{45}$$

$$(\rho_{f_2}^2 - m\rho)\frac{\partial \nu_z}{\partial t} = \phi\rho_{f_2}\frac{\eta_2}{\kappa}\nu_z - \phi\rho_{f_2}\frac{\eta_2}{\kappa}V_z + \rho_{f_2}\frac{\partial s}{\partial z} - m(\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z}), \tag{46}$$

$$(m\rho - \rho_{f_2}^2)\frac{\partial V_x}{\partial t} = \phi\rho\frac{\eta_2}{\kappa}\nu_x - \phi\rho\frac{\eta_2}{\kappa}V_x + \rho\frac{\partial s}{\partial x} - \rho_{f_2}(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z}), \tag{47}$$

$$(m\rho - \rho_{f_2}^2)\frac{\partial V_y}{\partial t} = \phi\rho\frac{\eta_2}{\kappa}\nu_y - \phi\rho\frac{\eta_2}{\kappa}V_y + \rho\frac{\partial s}{\partial y} - \rho_{f_2}(\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z}), \tag{48}$$

$$(m\rho - \rho_{f_2}^2)\frac{\partial V_z}{\partial t} = \phi\rho\frac{\eta_2}{\kappa}\nu_z - \phi\rho\frac{\eta_2}{\kappa}V_z + \rho\frac{\partial s}{\partial z} - \rho_{f_2}(\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z}). \tag{49}$$

Meanwhile, equations (37) - (43) can be written as

$$\frac{\partial \sigma_{xx}}{\partial t} = (H - 2\gamma D\phi)\frac{\partial v_x}{\partial x} + (H - 2\mu - 2\gamma D\phi)\frac{\partial v_y}{\partial y} + (H - 2\mu - 2\gamma D\phi)\frac{\partial v_z}{\partial z} + 2\gamma D\phi(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}), \tag{50}$$

$$\frac{\partial \sigma_{yy}}{\partial t} = (H - 2\mu - 2\gamma D\phi)\frac{\partial v_x}{\partial x} + (H - 2\gamma D\phi)\frac{\partial v_y}{\partial y} + (H - 2\mu - 2\gamma D\phi)\frac{\partial v_z}{\partial z} + 2\gamma D\phi(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}), \tag{51}$$

$$\frac{\partial \sigma_{zz}}{\partial t} = (H - 2\mu - 2\gamma D\phi)\frac{\partial v_x}{\partial x} + (H - 2\mu - 2\gamma D\phi)\frac{\partial v_y}{\partial y} + (H - 2\gamma D\phi)\frac{\partial v_z}{\partial z} + 2\gamma D\phi(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}), \tag{52}$$

$$\frac{\partial \sigma_{yz}}{\partial t} = \mu(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y}), \tag{53}$$

$$\frac{\partial \sigma_{xz}}{\partial t} = \mu(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}), \tag{54}$$

$$\frac{\partial \sigma_{xy}}{\partial t} = \mu(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}), \tag{55}$$

$$\frac{\partial s}{\partial t} = (2\gamma D - 2D\phi)(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}) + 2D\phi(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}). \tag{56}$$

In equations (44) – (56), the elastic coefficient H, which represents the modulus of plane wave, is complex in a modified patchy-saturated medium, since the bulk modulus K is complex in a patchy-saturated medium. The bulk modulus K is computed according to the bulk modulus Johnson (2001) established in a patchy-saturated medium, see our previous paper (Zhang and He, 2015). Therefore, we should solve the equations in the complex domain (see Appendix).

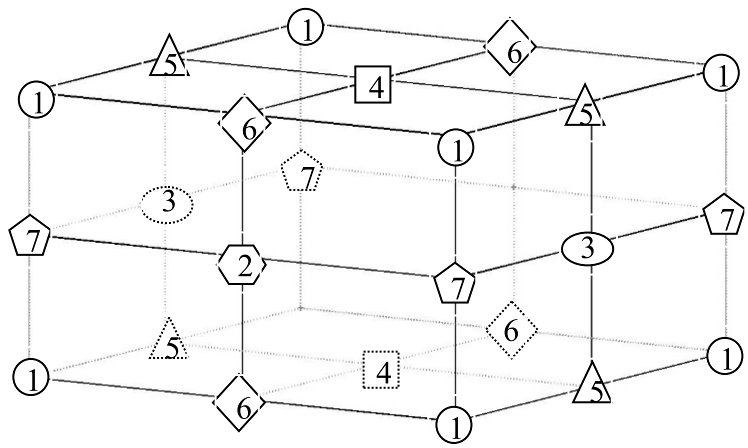

FIG. 2. Staggered- grid of modified patchy-saturated medium

Figure 2 is the staggered-grid we established. The position of each component in elastic wave field see Table 1.

Table 1. Positions of each component in elastic wave field

Staggered grid	1	2	3	4	5	6	7
Components of elastic wave field	σ_{xxre}, σ_{yyre}, σ_{zzre}, s_{re}, σ_{xxim}, σ_{yyim}, σ_{zzim}, s_{im}	σ_{yzre}, σ_{yzim}	σ_{xzre}, σ_{xzim}	σ_{xyre}, σ_{xyim}	v_{xre}, V_{xre}, v_{xim}, V_{xim}	v_{yre}, V_{yre}, v_{yim}, V_{yim}	v_{zre}, V_{zre}, v_{zim}, V_{zim}

Since we cannot compute unlimited regions, artificial boundary reflections inevitably exist, which will interfere with the wave field and affect our analysis. In order to absorb boundary reflections, the perfectly matched layer (PML) boundary conditions (Berenger, 1994) are used in this paper.

NUMERICAL SIMULATION OF MODIFIED PATCHY-SATURATED POROUS MEDIA

According to the parameters Johnson (2001) and Huang et al. (2012) used, we perform the numerical simulation of modified patchy-saturated porous media. The elastic parameters we used are in Table 2, in which the fluid 2 is oil. In Table 2, we introduce some parameters not previously introduced, which are used for computing the plane wave modulus H, referring to our previous paper (Zhang and He, 2015). Here, we just illustrate the physical meaning of the parameters. K_{f_1} and K_{f_2} are bulk moduli of fluid 1 and fluid 2 in a modified patchy-saturated medium, respectively; η_1 is the viscosity of fluid 1; K_s and K_b are the moduli of pure solid and skeleton with pores (i.e. dry solid), respectively; S (s is used instead in our previous paper) is a structure constant that depends on the pore structure and orientation.

Table 2. Elastic parameters in modified patchy-saturated media

Parameters	values	Parameters	values
ϕ	0.284	K_s	38×10^9 Pa
S_1	0.7	K_b	16×10^9 Pa
S_2	0.3	ρ_s	2650 $kg\cdot m^{-3}$
K_{f_1}	2.25×10^9 Pa	ρ_{f_1}	1000 $kg\cdot m^{-3}$
K_{f_2}	1.0×10^{10} Pa	ρ_{f_2}	800 $kg\cdot m^{-3}$
η_1	$1.0\times10^{-6} Pa\cdot s$	κ	$1.0\times10^{-13} m^2$
η_2	$1.0\times10^{-10} Pa\cdot s$	R_a	4.642cm
μ	14.61×10^9 Pa	R_b	10cm
S	2		

In this paper, we only study the wave propagation in isotropic homogeneous media. Therefore, a 2D homogeneous model is enough to test our computation method. The simulation results are shown in Figure 3. The model size we used here is 500×500 with 5m grid spacing. The source is shear wave source, which is put in the centre of the model. The time interval is 0.5ms and the wavelet dominant frequency is 40Hz.

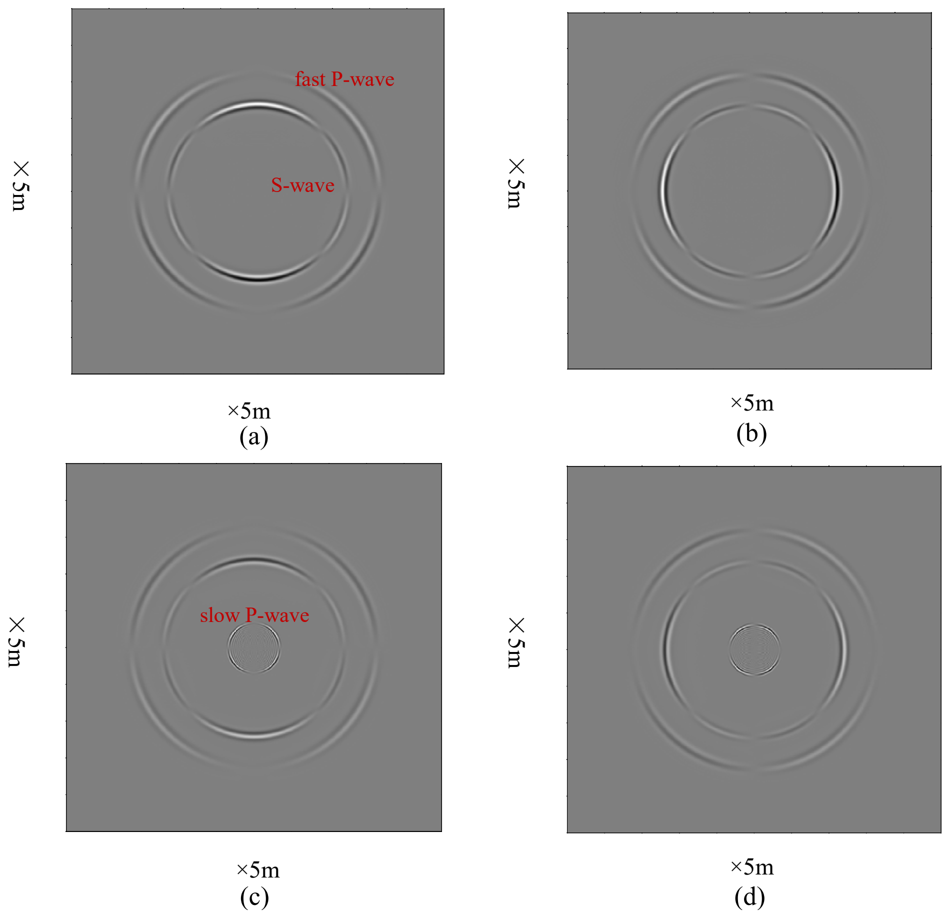

FIG. 3. The wave field snapshots of modified patchy-saturated porous medium (t=250ms): (a) v_x component, (b) v_z component, (c) V_x component, (d) V_z component

From Figure 3a and 3b, we can see that the fast P-wave and the S-wave are very clear while the slow P-wave is too weak to be seen. However, from Figure 3c and 3d, the two kinds of P-waves and S-wave can all be seen clearly and the velocities of the two P-waves and S-wave are equal to the theoretical results. Furthermore, we can see that the slow P-wave has a high dispersion which is also in accordance with the theoretical result. All these results illustrate that our numerical method is correct.

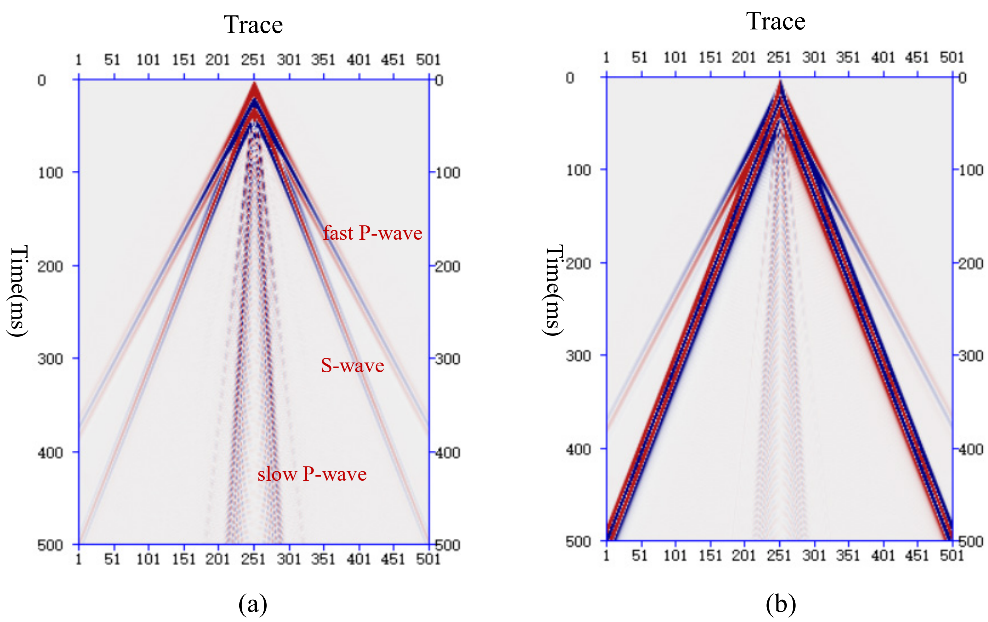

FIG .4. Synthetic seismograms of modified patchy-saturated model: (a) x component, (b) z component

Figure 4 is a synthetic seismogram of the modified patchy-saturated porous model. From Figure 4a and 4b, we can also see the three direct waves clearly which verifies our numerical modeling method.

Figure 5 is shown to illustrate our absorbing boundary conditions. Figure 5a is the snapshot of V_x component without absorbing boundary conditions, from which we can see a strong boundary reflection. Figure 5b is the snapshot of V_x component using PML absorbing boundary conditions. The boundary reflections are completely absorbed in Figure 5b which confirms our boundary condition's effectiveness.

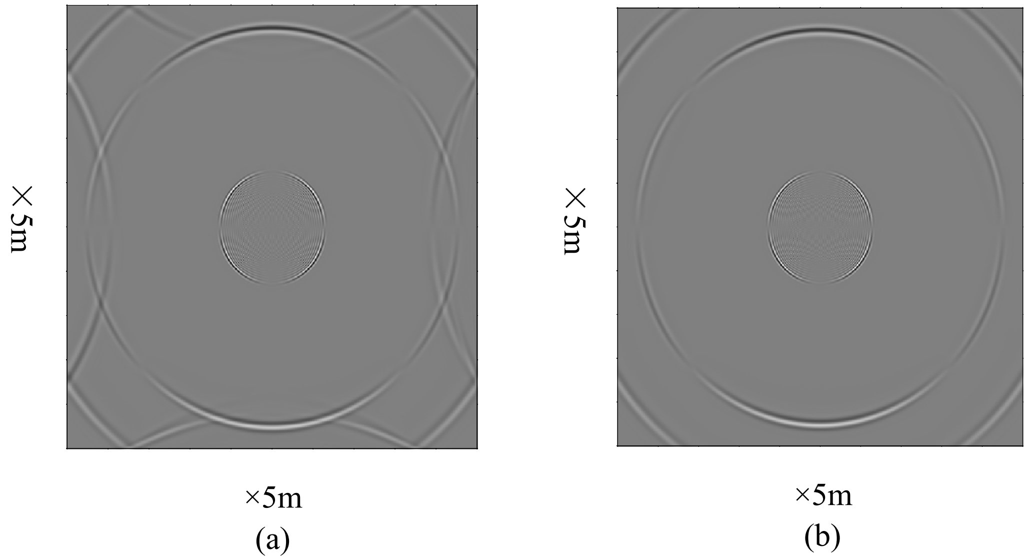

FIG. 5. Illustration of PML absorbing boundary conditions:(a) snapshot of V_x component without using any absorbing boundary conditions, (b) snapshot of V_x component after using PML absorbing boundary conditions.

CONCLUSIONS

When seismic energy propagates in underground media, especially in fluid-containing media, high attenuation and velocity dispersion will appear. Mesoscopic fluid flow in patchy-saturated media is the main cause for high seismic attenuation and velocity dissipation. However, the elastic wave equations in patchy saturation media have not been previously established. Based on the modified patchy-saturated porous model established in our previous paper, we derived stress-strain relations according to the method Biot used to construct elastic wave equations in porous media. After derivation, we obtained the dissipation function and kinetic energy in patchy-saturated porous media. By substituting the stress-strain relations and dissipation function and kinetic energy into the Lagrange's equations, we derived the elastic wave equations in modified patchy-saturated media.

For the purpose of examining our equations and studying wave propagations in partially saturated media, we perform numerical simulation of the elastic wave equations we established. In order to improve the computational precision and efficiency, we change the equations to first-order stress-velocity equations and deduced the 3D high-order finite difference schemes in the complex domain because of the complex bulk modulus in patchy saturated media. The PML boundary conditions are used to absorb the pseudo reflections from artificial boundaries. Numerical results show that: (1) There are two kinds of P-waves and one S-wave in patchy saturated media. (2) The energy of the slow P-wave is too weak to be seen in the "solid" phase of the patchy saturated porous media while it clearly exists in the fluid phase. (3) The high dispersion of the slow P-wave is apparent. (4) The fast P-wave and S-wave are obvious both in the solid phase and fluid phase. (5) The velocities of the three waves are consistent with the theoretical results. (6) The PML boundary conditions are effective in modified patchy-saturated porous media. All the above confirmed the correctness and effectiveness of our numerical method.

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APPENDIX

NUMERICAL SOLUTION OF ELASTIC WAVE EQUATIONS IN MODIFIED PATCHY-SATURATED MEDIA

Let $v_{xre}, v_{yre}, v_{zre}$ stand for the real parts of v_x, v_y and v_z, respectively; v_{xim}, v_{yim}, v_{zim} stand for the imaginary parts of v_x, v_y and v_z, respectively; V_{xre}, V_{yre}, V_{zre} represent for the real parts of V_x, V_y and V_z, respectively; V_{xim}, V_{yim}, V_{zim} represent the imaginary parts of V_x, V_y and V_z, respectively; $\sigma_{ijre}(i = x, y, \mathrm{z};\ j = x, y, \mathrm{z})$ and σ_{ijim} ($i = x, y, \mathrm{z}$; $j = x, y, \mathrm{z}$) are the real and imaginary parts of σ_{ij} ($i = x, y, \mathrm{z}$; $j = x, y, \mathrm{z}$), respectively; s_{re} and s_{im} are the real and imaginary parts of s, respectively; H_{re} and H_{im} represent for the real and imaginary parts of H, respectively. Then, equations (45) – (56) can be written as

$$
\begin{aligned}
&(\rho_{f_2}^2 - m\rho)\frac{\partial v_{xre}}{\partial t} \\
&\quad = \phi\rho_{f_2}\frac{\eta_2}{\kappa}v_{xre} - \phi\rho_{f_2}\frac{\eta_2}{\kappa}V_{xre} + \rho_{f_2}\frac{\partial s_{re}}{\partial x} - m\left(\frac{\partial\sigma_{xxre}}{\partial x} + \frac{\partial\sigma_{xyre}}{\partial y} + \frac{\partial\sigma_{xzre}}{\partial z}\right),
\end{aligned}
\tag{A-1}
$$

$$(\rho_{f_2}^2 - m\rho)\frac{\partial v_{yre}}{\partial t} = \phi\rho_{f_2}\frac{\eta_2}{\kappa}v_{yre} - \phi\rho_{f_2}\frac{\eta_2}{\kappa}V_{yre} + \rho_{f_2}\frac{\partial s_{re}}{\partial y} - m(\frac{\partial \sigma_{xyre}}{\partial x} + \frac{\partial \sigma_{yyre}}{\partial y} + \frac{\partial \sigma_{yzre}}{\partial z}), \tag{A-2}$$

$$(\rho_{f_2}^2 - m\rho)\frac{\partial v_{zre}}{\partial t} = \phi\rho_{f_2}\frac{\eta_2}{\kappa}v_{zre} - \phi\rho_{f_2}\frac{\eta_2}{\kappa}V_{zre} + \rho_{f_2}\frac{\partial s_{re}}{\partial z} - m(\frac{\partial \sigma_{xzre}}{\partial x} + \frac{\partial \sigma_{yzre}}{\partial y} + \frac{\partial \sigma_{zzre}}{\partial z}), \tag{A-3}$$

$$(m\rho - \rho_{f_2}^2)\frac{\partial V_{xre}}{\partial t} = \phi\rho\frac{\eta_2}{\kappa}v_{xre} - \phi\rho\frac{\eta_2}{\kappa}V_{xre} + \rho\frac{\partial s_{re}}{\partial x} - \rho_{f_2}(\frac{\partial \sigma_{xxre}}{\partial x} + \frac{\partial \sigma_{xyre}}{\partial y} + \frac{\partial \sigma_{xzre}}{\partial z}), \tag{A-4}$$

$$(m\rho - \rho_{f_2}^2)\frac{\partial V_{yre}}{\partial t} = \phi\rho\frac{\eta_2}{\kappa}v_{yre} - \phi\rho\frac{\eta_2}{\kappa}V_{yre} + \rho\frac{\partial s_{im}}{\partial y} - \rho_{f_2}(\frac{\partial \sigma_{xyre}}{\partial x} + \frac{\partial \sigma_{yyre}}{\partial y} + \frac{\partial \sigma_{yzre}}{\partial z}), \tag{A-5}$$

$$(m\rho - \rho_{f_2}^2)\frac{\partial V_{zre}}{\partial t} = \phi\rho\frac{\eta_2}{\kappa}v_{zre} - \phi\rho\frac{\eta_2}{\kappa}V_{zre} + \rho\frac{\partial s_{re}}{\partial z} - \rho_{f_2}(\frac{\partial \sigma_{xzre}}{\partial x} + \frac{\partial \sigma_{yzre}}{\partial y} + \frac{\partial \sigma_{zzre}}{\partial z}), \tag{A-6}$$

$$(\rho_{f_2}^2 - m\rho)\frac{\partial v_{xim}}{\partial t} = \phi\rho_{f_2}\frac{\eta_2}{\kappa}v_{xim} - \phi\rho_{f_2}\frac{\eta_2}{\kappa}V_{xim} + \rho_{f_2}\frac{\partial s_{im}}{\partial x} - m(\frac{\partial \sigma_{xxim}}{\partial x} + \frac{\partial \sigma_{xyim}}{\partial y} + \frac{\partial \sigma_{xzim}}{\partial z}), \tag{A-7}$$

$$(\rho_{f_2}^2 - m\rho)\frac{\partial v_{yim}}{\partial t} = \phi\rho_{f_2}\frac{\eta_2}{\kappa}v_{yim} - \phi\rho_{f_2}\frac{\eta_2}{\kappa}V_{yim} + \rho_{f_2}\frac{\partial s_{im}}{\partial y} - m(\frac{\partial \sigma_{xyim}}{\partial x} + \frac{\partial \sigma_{yyim}}{\partial y} + \frac{\partial \sigma_{yzim}}{\partial z}), \tag{A-8}$$

$$(\rho_{f_2}^2 - m\rho)\frac{\partial v_{zim}}{\partial t} = \phi\rho_{f_2}\frac{\eta_2}{\kappa}v_{zim} - \phi\rho_{f_2}\frac{\eta_2}{\kappa}V_{zim} + \rho_{f_2}\frac{\partial s_{im}}{\partial z} - m(\frac{\partial \sigma_{xzim}}{\partial x} + \frac{\partial \sigma_{yzim}}{\partial y} + \frac{\partial \sigma_{zzim}}{\partial z}), \tag{A-9}$$

$$
\begin{aligned}
&(m\rho-\rho_{f_2}^2)\frac{\partial V_{xim}}{\partial t}\\
&\quad=\phi\rho\frac{\eta_2}{\kappa}v_{xim}-\phi\rho\frac{\eta_2}{\kappa}V_{xim}+\rho\frac{\partial s_{im}}{\partial x}-\rho_{f_2}(\frac{\partial\sigma_{xxim}}{\partial x}+\frac{\partial\sigma_{xyim}}{\partial y}+\frac{\partial\sigma_{xzim}}{\partial z}),
\end{aligned}
\tag{A-10}
$$

$$
\begin{aligned}
&(m\rho-\rho_{f_2}^2)\frac{\partial V_{yim}}{\partial t}\\
&\quad=\phi\rho\frac{\eta_2}{\kappa}v_{yim}-\phi\rho\frac{\eta_2}{\kappa}V_{yim}+\rho\frac{\partial s_{im}}{\partial y}-\rho_{f_2}(\frac{\partial\sigma_{xyim}}{\partial x}+\frac{\partial\sigma_{yyim}}{\partial y}+\frac{\partial\sigma_{yzim}}{\partial z}),
\end{aligned}
\tag{A-11}
$$

$$
\begin{aligned}
&(m\rho-\rho_{f_2}^2)\frac{\partial V_{zim}}{\partial t}\\
&\quad=\phi\rho\frac{\eta_2}{\kappa}v_{zim}-\phi\rho\frac{\eta_2}{\kappa}V_{zim}+\rho\frac{\partial s_{im}}{\partial z}-\rho_{f_2}(\frac{\partial\sigma_{xzim}}{\partial x}+\frac{\partial\sigma_{yzim}}{\partial y}+\frac{\partial\sigma_{zzim}}{\partial z}),
\end{aligned}
\tag{A-12}
$$

$$
\begin{aligned}
\frac{\partial\sigma_{xxre}}{\partial t}&=(H_{re}-2\gamma D\phi)\frac{\partial v_{xre}}{\partial x}-H_{im}\frac{\partial v_{xim}}{\partial x}\\
&\quad+(H_{re}-2\mu-2\gamma D\phi)\frac{\partial v_{yre}}{\partial y}-H_{im}\frac{\partial v_{yim}}{\partial y}+(\mathrm{H}_{re}-2\mu-2\gamma\,\mathrm{D}\,\phi)\frac{\partial v_{zre}}{\partial z}\\
&\quad-H_{im}\frac{\partial v_{zim}}{\partial z}+2\gamma\,\mathrm{D}\,\phi(\frac{\partial V_{xre}}{\partial x}+\frac{\partial V_{yre}}{\partial y}+\frac{\partial V_{zre}}{\partial z}),
\end{aligned}
\tag{A-13}
$$

$$
\begin{aligned}
\frac{\partial\sigma_{yyre}}{\partial t}&=(H_{re}-2\mu-2\gamma D\phi)\frac{\partial v_{xre}}{\partial x}-H_{im}\frac{\partial v_{xim}}{\partial x}\\
&\quad+(H_{re}-2\gamma D\phi)\frac{\partial v_{yre}}{\partial y}-H_{im}\frac{\partial v_{yim}}{\partial y}+(H_{re}-2\mu-2\gamma D\phi)\frac{\partial v_{zre}}{\partial z}\\
&\quad-H_{im}\frac{\partial v_{zim}}{\partial z}+2\gamma D\phi(\frac{\partial V_{xre}}{\partial x}+\frac{\partial V_{yre}}{\partial y}+\frac{\partial V_{zre}}{\partial z}),
\end{aligned}
\tag{A-14}
$$

$$
\begin{aligned}
\frac{\partial\sigma_{zzre}}{\partial t}&=(H_{re}-2\mu-2\gamma D\phi)\frac{\partial v_{xre}}{\partial x}-H_{im}\frac{\partial v_{xim}}{\partial x}\\
&\quad+(H_{re}-2\mu-2\gamma D\phi)\frac{\partial v_{yre}}{\partial y}-H_{im}\frac{\partial v_{yim}}{\partial y}+(H_{re}-2\gamma D\phi)\frac{\partial v_{zre}}{\partial z}\\
&\quad-H_{im}\frac{\partial v_{zim}}{\partial z}+2\gamma D\phi(\frac{\partial V_{xre}}{\partial x}+\frac{\partial V_{yre}}{\partial y}+\frac{\partial V_{zre}}{\partial z}),
\end{aligned}
\tag{A-15}
$$

$$
\frac{\partial\sigma_{yzre}}{\partial t}=\mu(\frac{\partial v_{yre}}{\partial z}+\frac{\partial v_{zre}}{\partial y}),
\tag{A-16}
$$

$$
\frac{\partial\sigma_{xzre}}{\partial t}=\mu(\frac{\partial v_{xre}}{\partial z}+\frac{\partial v_{zre}}{\partial x}),
\tag{A-17}
$$

$$\frac{\partial \sigma_{xyre}}{\partial t} = \mu\left(\frac{\partial v_{xre}}{\partial y} + \frac{\partial v_{yre}}{\partial x}\right), \tag{A-18}$$

$$\frac{\partial s_{re}}{\partial t} = (2\gamma D - 2D\phi)\left(\frac{\partial v_{xre}}{\partial x} + \frac{\partial v_{yre}}{\partial y} + \frac{\partial v_{zre}}{\partial z}\right) + 2D\phi\left(\frac{\partial V_{xre}}{\partial x} + \frac{\partial V_{yre}}{\partial y} + \frac{\partial V_{zre}}{\partial z}\right), \tag{A-19}$$

$$\begin{aligned} \frac{\partial \sigma_{xxim}}{\partial t} &= (H_{re} - 2\gamma D\phi)\frac{\partial v_{xim}}{\partial x} + H_{im}\frac{\partial v_{xre}}{\partial x} \\ &+ (H_{re} - 2\mu - 2\gamma D\phi)\frac{\partial v_{yim}}{\partial y} + H_{im}\frac{\partial v_{yre}}{\partial y} + (H_{re} - 2\mu - 2\gamma D\phi)\frac{\partial v_{zim}}{\partial z} \\ &+ H_{im}\frac{\partial v_{zre}}{\partial z} + 2\gamma D\phi\left(\frac{\partial V_{xim}}{\partial x} + \frac{\partial V_{yim}}{\partial y} + \frac{\partial V_{zim}}{\partial z}\right), \end{aligned} \tag{A-20}$$

$$\begin{aligned} \frac{\partial \sigma_{yyim}}{\partial t} &= (H_{re} - 2\mu - 2\gamma D\phi)\frac{\partial v_{xim}}{\partial x} + H_{im}\frac{\partial v_{xre}}{\partial x} \\ &+ (H_{re} - 2\gamma D\phi)\frac{\partial v_{yim}}{\partial y} + H_{im}\frac{\partial v_{yre}}{\partial y} + (H_{re} - 2\mu - 2\gamma D\phi)\frac{\partial v_{zim}}{\partial z} \\ &+ H_{im}\frac{\partial v_{zre}}{\partial z} + 2\gamma D\phi\left(\frac{\partial V_{xim}}{\partial x} + \frac{\partial V_{yim}}{\partial y} + \frac{\partial V_{zim}}{\partial z}\right), \end{aligned} \tag{A-21}$$

$$\begin{aligned} \frac{\partial \sigma_{zzim}}{\partial t} &= (H_{re} - 2\mu - 2\gamma D\phi)\frac{\partial v_{xim}}{\partial x} + H_{im}\frac{\partial v_{xre}}{\partial x} \\ &+ (H_{re} - 2\mu - 2\gamma D\phi)\frac{\partial v_{yim}}{\partial y} + H_{im}\frac{\partial v_{yre}}{\partial y} + (H_{re} - 2\gamma D\phi)\frac{\partial v_{zim}}{\partial z} \\ &+ H_{im}\frac{\partial v_{zre}}{\partial z} + 2\gamma D\phi\left(\frac{\partial V_{xim}}{\partial x} + \frac{\partial V_{yim}}{\partial y} + \frac{\partial V_{zim}}{\partial z}\right), \end{aligned} \tag{A-22}$$

$$\frac{\partial \sigma_{yzim}}{\partial t} = \mu\left(\frac{\partial v_{yim}}{\partial z} + \frac{\partial v_{zim}}{\partial y}\right), \tag{A-23}$$

$$\frac{\partial \sigma_{xzim}}{\partial t} = \mu\left(\frac{\partial v_{xim}}{\partial z} + \frac{\partial v_{zim}}{\partial x}\right), \tag{A-24}$$

$$\frac{\partial \sigma_{xyim}}{\partial t} = \mu\left(\frac{\partial v_{xim}}{\partial y} + \frac{\partial v_{yim}}{\partial x}\right), \tag{A-25}$$

$$\frac{\partial s_{im}}{\partial t} = (2\gamma D - 2D\phi)\left(\frac{\partial v_{xim}}{\partial x} + \frac{\partial v_{yim}}{\partial y} + \frac{\partial v_{zim}}{\partial z}\right) + 2D\phi\left(\frac{\partial V_{xim}}{\partial x} + \frac{\partial V_{yim}}{\partial y} + \frac{\partial V_{zim}}{\partial z}\right). \tag{A-26}$$

According to equations (A-1) – (A-26), we derive the 3D high-order finite difference schemes as follows

$$
\begin{aligned}
v_{xre}^{n+1}\left(i^{+},\mathrm{j},\mathrm{k}\right) = {} & v_{xre}^{n}\left(i^{+},\mathrm{j},\mathrm{k}\right)[1+D_5\left(i^{+},\mathrm{j},\mathrm{k}\right)b_1\Delta\mathrm{t}] \\
& -V_{xre}\left(i^{+},\mathrm{j},\mathrm{k}\right)\times D_5\left(i^{+},\mathrm{j},\mathrm{k}\right)b_1\Delta\mathrm{t} \\
& +v_{yre}^{n}\left(i,\mathrm{j}^{+},\mathrm{k}\right)D_5\left(i^{+},\mathrm{j},\mathrm{k}\right)b_2\Delta\mathrm{t} \\
& -V_{yre}^{n}\left(i,\mathrm{j}^{+},\mathrm{k}\right)D_5\left(i^{+},\mathrm{j},\mathrm{k}\right)b_2\Delta\mathrm{t} \\
& +v_{zre}^{n}\left(i,\mathrm{j},\mathrm{k}^{+}\right)D_5\left(i^{+},\mathrm{j},\mathrm{k}\right)b_3\Delta\mathrm{t} \\
& -V_{zre}^{n}\left(i,\mathrm{j},\mathrm{k}^{+}\right)D_5\left(i^{+},\mathrm{j},\mathrm{k}\right)b_3\mathrm{t} \\
& +D_3\left(i^{+},\mathrm{j},\mathrm{k}\right)L_x^{+}\left[s_{re}\left(i,\mathrm{j},\mathrm{k}\right)\right]\Delta\mathrm{t}-D_2\left(i^{+},\mathrm{j},\mathrm{k}\right) \\
& \times\left\{L_x^{+}\left[\sigma_{xxre}^{n+}\left(i,\mathrm{j},\mathrm{k}\right)\right]+L_y^{-}\left[\sigma_{xyre}^{n+}\left(i^{+},j^{+},\mathrm{k}\right)\right]+L_z^{-}\left[\sigma_{xzre}^{n+}\left(i^{+},\mathrm{j},k^{+}\right)\right]\right\}\Delta\mathrm{t},
\end{aligned}
\tag{A-27}
$$

$$
\begin{aligned}
V_{xre}^{n+1}\left(i^{+},\mathrm{j},\mathrm{k}\right) = {} & V_{xre}^{n}\left(i^{+},\mathrm{j},\mathrm{k}\right)[1+D_4\left(i^{+},\mathrm{j},\mathrm{k}\right)b_1\Delta\mathrm{t}] \\
& -v_{xre}\left(i^{+},\mathrm{j},\mathrm{k}\right)\times D_4\left(i^{+},\mathrm{j},\mathrm{k}\right)b_1\Delta\mathrm{t} \\
& +V_{yre}^{n}\left(i,\mathrm{j}^{+},\mathrm{k}\right)D_4\left(i^{+},\mathrm{j},\mathrm{k}\right)b_2\Delta\mathrm{t} \\
& -v_{yre}^{n}\left(i,\mathrm{j}^{+},\mathrm{k}\right)D_4\left(i^{+},\mathrm{j},\mathrm{k}\right)b_2\Delta\mathrm{t} \\
& +V_{zre}^{n}\left(i,\mathrm{j},\mathrm{k}^{+}\right)D_4\left(i^{+},\mathrm{j},\mathrm{k}\right)b_3\Delta\mathrm{t} \\
& -v_{zre}^{n}\left(i,\mathrm{j},\mathrm{k}^{+}\right)D_4\left(i^{+},\mathrm{j},\mathrm{k}\right)b_3\Delta\mathrm{t} \\
& -D_1\left(i^{+},\mathrm{j},\mathrm{k}\right)L_x^{+}\left[s_{re}\left(i,\mathrm{j},\mathrm{k}\right)\right]\Delta\mathrm{t}+D_3\left(i^{+},\mathrm{j},\mathrm{k}\right) \\
& \times\left\{L_x^{+}\left[\sigma_{xxre}^{n+}\left(i,\mathrm{j},\mathrm{k}\right)\right]+L_y^{-}\left[\sigma_{xyre}^{n+}\left(i^{+},j^{+},\mathrm{k}\right)\right]+L_z^{-}\left[\sigma_{xzre}^{n+}\left(i^{+},\mathrm{j},k^{+}\right)\right]\right\}\Delta\mathrm{t},
\end{aligned}
\tag{A-28}
$$

$$
\begin{aligned}
\sigma_{xxre}^{n+}\left(i,\mathrm{j},\mathrm{k}\right) = {} & \sigma_{xxre}^{n-}\left(i,\mathrm{j},\mathrm{k}\right)+C_{11}\left(i,j,k\right)L_x^{-}\left[v_{xre}^{n}\left(i^{+},\mathrm{j},\mathrm{k}\right)\right]\Delta\mathrm{t} \\
& -H_{im}\left(i,j,k\right)L_x^{-}\left[v_{xim}^{n}\left(i^{+},\mathrm{j},\mathrm{k}\right)\right]\Delta\mathrm{t} \\
& +C_{12}\left(i,j,k\right)L_y^{-}\left[v_{yre}^{n}\left(i,j^{+},\mathrm{k}\right)\right]\Delta\mathrm{t} \\
& -H_{im}\left(i,j,k\right)L_y^{-}\left[v_{yim}^{n}\left(i,j^{+},\mathrm{k}\right)\right]\Delta\mathrm{t} \\
& +C_{13}\left(i,j,k\right)L_z^{-}\left[v_{zre}^{n}\left(i,\mathrm{j},k^{+}\right)\right]\Delta\mathrm{t} \\
& -H_{im}\left(i,j,k\right)L_z^{-}\left[v_{zim}^{n}\left(i,\mathrm{j},k^{+}\right)\right]\Delta\mathrm{t} \\
& +C_{14}\left(i,j,k\right)\{L_z^{-}\left[v_{yre}^{n}\left(i,j^{+},\mathrm{k}\right)\right]+L_y^{-}\left[v_{zre}^{n}\left(i,j,\mathrm{k}^{+}\right)\right]\}\Delta\mathrm{t} \\
& +C_{15}\left(i,j,k\right)\{L_z^{-}\left[v_{xre}^{n}\left(i^{+},j,\mathrm{k}\right)\right]+L_x^{-}\left[v_{zre}^{n}\left(i,j,\mathrm{k}^{+}\right)\right]\}\Delta\mathrm{t} \\
& +C_{16}\left(i,j,k\right)\{L_y^{-}\left[v_{xre}^{n}\left(i^{+},j,\mathrm{k}\right)\right]+L_x^{-}\left[v_{yre}^{n}\left(i,j^{+},\mathrm{k}\right)\right]\}\Delta\mathrm{t} \\
& +Q_1(i,j,k)\left\{L_x^{-}\left[V_{xre}\left(i^{+},\mathrm{j},\mathrm{k}\right)\right]+L_y^{-}\left[V_{yre}\left(i,j^{+},\mathrm{k}\right)\right]+L_z^{-}\left[V_{zre}\left(i,\mathrm{j},k^{+}\right)\right]\right\},
\end{aligned}
\tag{A-29}
$$

$$
\begin{aligned}
\sigma_{xzre}^{n+}\left(i^{+},\mathrm{j},k^{+}\right)=&\ \sigma_{xzre}^{n-}\left(i^{+},\mathrm{j},k^{+}\right)+C_{15}\left(i^{+},\mathrm{j},k\right)L_{x}^{-}\left[v_{xre}^{n}\left(i^{+},\mathrm{j},\mathrm{k}\right)\right]\Delta\mathrm{t}\\
&+C_{25}\left(i,j^{+},\mathrm{k}\right)L_{y}^{-}\left[v_{yre}^{n}\left(i,j^{+},\mathrm{k}\right)\right]\Delta\mathrm{t}\\
&+C_{35}\left(i,\mathrm{j},k^{+}\right)L_{z}^{-}\left[v_{zre}^{n}\left(i,\mathrm{j},k^{+}\right)\right]\Delta\mathrm{t}\\
&+C_{45}\left(i,j^{+},k^{+}\right)\{L_{z}^{-}\left[v_{yre}^{n}\left(i,j^{+},\mathrm{k}\right)\right]+L_{y}^{-}\left[v_{zre}^{n}\left(i,j,\mathrm{k}^{+}\right)\right]\}\Delta\mathrm{t}\\
&+C_{55}\left(i^{+},\mathrm{j},k^{+}\right)\left\{L_{z}^{+}\left[v_{xre}^{n}\left(i^{+},\mathrm{j},\mathrm{k}\right)\right]+L_{x}^{+}\left[v_{zre}^{n}\left(i,\mathrm{j},k^{+}\right)\right]\right\}\Delta\mathrm{t}\\
&+C_{56}\left(i^{+},j^{+},k\right)\{L_{y}^{-}\left[v_{xre}^{n}\left(i^{+},j,\mathrm{k}\right)\right]+L_{x}^{-}\left[v_{yre}^{n}\left(i,j^{+},\mathrm{k}\right)\right]\}\Delta\mathrm{t}\\
&+Q_{5}\left(i^{+},j,k^{+}\right)\left\{\begin{aligned}&L_{x}^{-}\left[V_{xre}\left(i^{+},\mathrm{j},\mathrm{k}\right)\right]\\&+L_{y}^{-}\left[V_{yre}\left(i,j^{+},\mathrm{k}\right)\right]+L_{z}^{-}\left[V_{zre}\left(i,\mathrm{j},k^{+}\right)\right]\end{aligned}\right\},
\end{aligned}
\tag{A-30}
$$

$$
\begin{aligned}
s_{re}^{n+}\left(i,\mathrm{j},\mathrm{k}\right)=&\ s_{re}^{n-}\left(i,\mathrm{j},\mathrm{k}\right)+[Q_{1}\left(i,j,k\right)/\phi(i,j,k)-R(i,j,k)]L_{x}^{-}\left[v_{x}\left(i^{+},\mathrm{j},\mathrm{k}\right)\right]\Delta\mathrm{t}\\
&+[Q_{2}\left(i,j,k\right)/\phi(i,j,k)-R(i,j,k)]L_{y}^{-}\left[v_{yre}\left(i,j^{+},\mathrm{k}\right)\right]\Delta\mathrm{t}\\
&+[Q_{3}\left(i,j,k\right)/\phi(i,j,k)-R(i,j,k)]L_{z}^{-}\left[v_{z}\left(i,\mathrm{j},k^{+}\right)\right]\Delta\mathrm{t}\\
&+Q_{4}(i,j^{+},k^{+})\{L_{z}^{-}\left[v_{yre}^{n}\left(i,j^{+},\mathrm{k}\right)\right]+L_{y}^{-}\left[v_{zre}^{n}\left(i,j,\mathrm{k}^{+}\right)\right]\}\Delta\mathrm{t}\\
&+Q_{5}(i^{+},\mathrm{j},\mathrm{k}^{+})\left\{L_{z}^{+}\left[v_{xre}^{n}\left(i^{+},\mathrm{j},\mathrm{k}\right)\right]+L_{x}^{+}\left[v_{zre}^{n}\left(i,\mathrm{j},k^{+}\right)\right]\right\}\Delta\mathrm{t}\\
&+Q_{6}(i^{+},\mathrm{j},\mathrm{k}^{+})\{L_{y}^{-}\left[v_{xre}^{n}\left(i^{+},j,\mathrm{k}\right)\right]+L_{x}^{-}\left[v_{yre}^{n}\left(i,j^{+},\mathrm{k}\right)\right]\}\Delta\mathrm{t}\\
&+R\left(i,j,k\right)\left\{\begin{aligned}&L_{x}^{-}\left[V_{xre}\left(i^{+},\mathrm{j},\mathrm{k}\right)\right]\\&+L_{y}^{-}\left[V_{yre}\left(i,j^{+},\mathrm{k}\right)\right]+L_{z}^{-}\left[V_{zre}\left(i,\mathrm{j},k^{+}\right)\right]\end{aligned}\right\}\Delta\mathrm{t},
\end{aligned}
\tag{A-31}
$$

$$
\begin{aligned}
v_{xim}^{n+1}\left(i^{+},\mathrm{j},\mathrm{k}\right)=&\ v_{xim}^{n}\left(i^{+},\mathrm{j},\mathrm{k}\right)[1+D_{5}\left(i^{+},\mathrm{j},\mathrm{k}\right)b_{1}\Delta\mathrm{t}]\\
&-V_{xim}\left(i^{+},\mathrm{j},\mathrm{k}\right)\times D_{5}\left(i^{+},\mathrm{j},\mathrm{k}\right)b_{1}\Delta\mathrm{t}\\
&+v_{yim}^{n}\left(i,\mathrm{j}^{+},\mathrm{k}\right)D_{5}\left(i^{+},\mathrm{j},\mathrm{k}\right)b_{2}\Delta\mathrm{t}\\
&-V_{yim}^{n}\left(i,\mathrm{j}^{+},\mathrm{k}\right)D_{5}\left(i^{+},\mathrm{j},\mathrm{k}\right)b_{2}\Delta\mathrm{t}\\
&+v_{zim}^{n}\left(i,\mathrm{j},\mathrm{k}^{+}\right)D_{5}\left(i^{+},\mathrm{j},\mathrm{k}\right)b_{3}\Delta\mathrm{t}\\
&-V_{zim}^{n}\left(i,\mathrm{j},\mathrm{k}^{+}\right)D_{5}\left(i^{+},\mathrm{j},\mathrm{k}\right)b_{3}\mathrm{t}\\
&+D_{3}\left(i^{+},\mathrm{j},\mathrm{k}\right)L_{x}^{+}\left[s_{im}\left(i,\mathrm{j},\mathrm{k}\right)\right]\Delta\mathrm{t}-D_{2}\left(i^{+},\mathrm{j},\mathrm{k}\right)\\
&\times\left\{L_{x}^{+}\left[\sigma_{xxim}^{n+}\left(i,\mathrm{j},\mathrm{k}\right)\right]+L_{y}^{-}\left[\sigma_{xyim}^{n+}\left(i^{+},j^{+},\mathrm{k}\right)\right]+L_{z}^{-}\left[\sigma_{xzim}^{n+}\left(i^{+},\mathrm{j},k^{+}\right)\right]\right\}\Delta\mathrm{t},
\end{aligned}
\tag{A-32}
$$

$$
\begin{aligned}
V_{xim}^{n+1}\left(i^{+},\mathrm{j},\mathrm{k}\right) = V_{xim}^{n}\left(i^{+},\mathrm{j},\mathrm{k}\right)&[1+D_4\left(i^{+},\mathrm{j},\mathrm{k}\right)b_1\Delta\mathrm{t}] \\
&-v_{xim}\left(i^{+},\mathrm{j},\mathrm{k}\right)\times D_4\left(i^{+},\mathrm{j},\mathrm{k}\right)b_1\Delta\mathrm{t} \\
&+V_{yim}^{n}\left(i,\mathrm{j}^{+},\mathrm{k}\right)D_4\left(i^{+},\mathrm{j},\mathrm{k}\right)b_2\Delta\mathrm{t} \\
&-v_{yim}^{n}\left(i,\mathrm{j}^{+},\mathrm{k}\right)D_4\left(i^{+},\mathrm{j},\mathrm{k}\right)b_2\Delta\mathrm{t} \\
&+V_{zim}^{n}\left(i,\mathrm{j},\mathrm{k}^{+}\right)D_4\left(i^{+},\mathrm{j},\mathrm{k}\right)b_3\Delta\mathrm{t} \\
&-v_{zim}^{n}\left(i,\mathrm{j},\mathrm{k}^{+}\right)D_4\left(i^{+},\mathrm{j},\mathrm{k}\right)b_3\Delta\mathrm{t} \\
&-D_1\left(i^{+},\mathrm{j},\mathrm{k}\right)L_x^{+}\left[s_{im}\left(i,\mathrm{j},\mathrm{k}\right)\right]\Delta\mathrm{t}+D_3\left(i^{+},\mathrm{j},\mathrm{k}\right) \\
&\times\left\{L_x^{+}\left[\sigma_{xxim}^{n+}\left(i,\mathrm{j},\mathrm{k}\right)\right]+L_y^{-}\left[\sigma_{xyim}^{n+}\left(i^{+},j^{+},\mathrm{k}\right)\right]+L_z^{-}\left[\sigma_{xzim}^{n+}\left(i^{+},\mathrm{j},k^{+}\right)\right]\right\}\Delta\mathrm{t},
\end{aligned}
\tag{A-33}
$$

$$
\begin{aligned}
\sigma_{xxim}^{n+}\left(i,\mathrm{j},\mathrm{k}\right) = \sigma_{xxim}^{n-}\left(i,\mathrm{j},\mathrm{k}\right)&+C_{11}\left(i,j,k\right)L_x^{-}\left[v_{xim}^{n}\left(i^{+},\mathrm{j},\mathrm{k}\right)\right]\Delta\mathrm{t} \\
&+H_{im}\left(i,j,k\right)L_x^{-}\left[v_{xre}^{n}\left(i^{+},\mathrm{j},\mathrm{k}\right)\right]\Delta\mathrm{t} \\
&+C_{12}\left(i,j,k\right)L_y^{-}\left[v_{yim}^{n}\left(i,j^{+},\mathrm{k}\right)\right]\Delta\mathrm{t} \\
&+H_{im}\left(i,j,k\right)L_y^{-}\left[v_{yre}^{n}\left(i,j^{+},\mathrm{k}\right)\right]\Delta\mathrm{t} \\
&+C_{13}\left(i,j,k\right)L_z^{-}\left[v_{zim}^{n}\left(i,\mathrm{j},k^{+}\right)\right]\Delta\mathrm{t} \\
&+H_{im}\left(i,j,k\right)L_z^{-}\left[v_{zre}^{n}\left(i,\mathrm{j},k^{+}\right)\right]\Delta\mathrm{t} \\
&+C_{14}\left(i,j,k\right)\{L_z^{-}\left[v_{yim}^{n}\left(i,j^{+},\mathrm{k}\right)\right]+L_y^{-}\left[v_{zim}^{n}\left(i,j,\mathrm{k}^{+}\right)\right]\}\Delta\mathrm{t} \\
&+C_{15}\left(i,j,k\right)\{L_z^{-}\left[v_{xim}^{n}\left(i^{+},j,\mathrm{k}\right)\right]+L_x^{-}\left[v_{zim}^{n}\left(i,j,\mathrm{k}^{+}\right)\right]\}\Delta\mathrm{t} \\
&+C_{16}\left(i,j,k\right)\{L_y^{-}\left[v_{xim}^{n}\left(i^{+},j,\mathrm{k}\right)\right]+L_x^{-}\left[v_{yim}^{n}\left(i,j^{+},\mathrm{k}\right)\right]\}\Delta\mathrm{t} \\
&+Q_1\left(i,j,k\right)\left\{L_x^{-}\left[V_{xim}\left(i^{+},\mathrm{j},\mathrm{k}\right)\right]+L_y^{-}\left[V_{yim}\left(i,j^{+},\mathrm{k}\right)\right]+L_z^{-}\left[V_{zim}\left(i,\mathrm{j},k^{+}\right)\right]\right\},
\end{aligned}
\tag{A-34}
$$

$$
\begin{aligned}
\sigma_{xzim}^{n+}\left(i^{+},\mathrm{j},k^{+}\right) = \sigma_{xzim}^{n-}\left(i^{+},\mathrm{j},k^{+}\right)&+C_{15}\left(i^{+},\mathrm{j},k\right)L_x^{-}\left[v_{xim}^{n}\left(i^{+},\mathrm{j},\mathrm{k}\right)\right]\Delta\mathrm{t} \\
&+C_{25}\left(i,j^{+},\mathrm{k}\right)L_y^{-}\left[v_{yim}^{n}\left(i,j^{+},\mathrm{k}\right)\right]\Delta\mathrm{t} \\
&+C_{35}\left(i,\mathrm{j},k^{+}\right)L_z^{-}\left[v_{zim}^{n}\left(i,\mathrm{j},k^{+}\right)\right]\Delta\mathrm{t} \\
&+C_{45}\left(i,j^{+},k^{+}\right)\{L_z^{-}\left[v_{yim}^{n}\left(i,j^{+},\mathrm{k}\right)\right]+L_y^{-}\left[v_{zim}^{n}\left(i,j,\mathrm{k}^{+}\right)\right]\}\Delta\mathrm{t} \\
&+C_{55}\left(i^{+},\mathrm{j},k^{+}\right)\left\{L_z^{+}\left[v_{xim}^{n}\left(i^{+},\mathrm{j},\mathrm{k}\right)\right]+L_x^{+}\left[v_{zim}^{n}\left(i,\mathrm{j},k^{+}\right)\right]\right\}\Delta\mathrm{t} \\
&+C_{56}\left(i^{+},j^{+},k\right)\{L_y^{-}\left[v_{xim}^{n}\left(i^{+},j,\mathrm{k}\right)\right]+L_x^{-}\left[v_{yim}^{n}\left(i,j^{+},\mathrm{k}\right)\right]\}\Delta\mathrm{t} \\
&+Q_5\left(i^{+},j,k^{+}\right)\left\{L_x^{-}\left[V_{xim}\left(i^{+},\mathrm{j},\mathrm{k}\right)\right]+L_y^{-}\left[V_{yim}\left(i,j^{+},\mathrm{k}\right)\right]+L_z^{-}\left[V_{zim}\left(i,\mathrm{j},k^{+}\right)\right]\right\},
\end{aligned}
\tag{A-35}
$$

$$
\begin{aligned}
S_{im}^{n+}(i,\mathrm{j},\mathrm{k}) &= S_{im}^{n-}(i,\mathrm{j},\mathrm{k}) + [Q_1(i,j,k)/\phi(i,j,k) - R(i,j,k)]L_x^-\left[v_{xim}\left(i^+,\mathrm{j},\mathrm{k}\right)\right]\Delta\mathrm{t} \\
&+ [Q_2(i,j,k)/\phi(i,j,k) - R(i,j,k)]L_y^-\left[v_{yim}\left(i,j^+,\mathrm{k}\right)\right]\Delta\mathrm{t} \\
&+ [Q_3(i,j,k)/\phi(i,j,k) - R(i,j,k)]L_z^-\left[v_{zim}\left(i,\mathrm{j},k^+\right)\right]\Delta\mathrm{t} \\
&+ Q_4(i,j^+,k^+)\{L_z^-\left[v_{yim}^n\left(i,j^+,\mathrm{k}\right)\right] + L_y^-\left[v_{zim}^n\left(i,j,\mathrm{k}^+\right)\right]\}\Delta\mathrm{t} \\
&+ Q_5(i^+,\mathrm{j},\mathrm{k}^+)\left\{L_z^+\left[v_{xim}^n\left(i^+,\mathrm{j},\mathrm{k}\right)\right] + L_x^+\left[v_{zim}^n\left(i,\mathrm{j},k^+\right)\right]\right\}\Delta\mathrm{t} \\
&+ Q_6(i^+,\mathrm{j},\mathrm{k}^+)\{L_y^-\left[v_{xim}^n\left(i^+,j,\mathrm{k}\right)\right] + L_x^-\left[v_{yim}^n\left(i,j^+,\mathrm{k}\right)\right]\}\Delta\mathrm{t} \\
&+ R(i,j,k)\left\{\begin{matrix} L_x^-\left[V_{xim}\left(i^+,\mathrm{j},\mathrm{k}\right)\right] + L_y^-\left[V_{yim}\left(i,j^+,\mathrm{k}\right)\right] \\ +L_z^-\left[V_{zim}\left(i,\mathrm{j},k^+\right)\right] \end{matrix}\right\}\Delta\mathrm{t},
\end{aligned}
\tag{A-36}
$$

with

$$
D_1 = \frac{\rho}{\rho_{f_2}^2 - m\rho}, \quad D_2 = \frac{\mathrm{m}}{\rho_{f_2}^2 - m\rho}, \quad D_3 = \frac{\rho_{f_2}}{\rho_{f_2}^2 - m\rho}, \quad D_4 = \frac{\phi\rho\,\eta_2/\kappa}{\rho_{f_2}^2 - m\rho}, \quad D_5 = \frac{\phi\rho_{f_2}\,\eta_2/\kappa}{\rho_{f_2}^2 - m\rho},
$$

$$
b_1 = 1,\ b_2 = b_3 = 0,\ C_{11} = H_{re} - 2\gamma D\phi,\ C_{12} = C_{13} = H_{re} - 2\mu - 2\gamma D\phi,
$$

$$
C_{14} = C_{15} = C_{16} = C_{25} = C_{35} = C_{45} = C_{56} = 0, \quad C_{55} = \mu,
$$

$$
Q_1 = Q_2 = Q_3 = 2\gamma D\phi, \quad Q_4 = Q_5 = Q_6 = 0, \quad R = 2D\phi,
$$

$$
\mathrm{L}_{\mathrm{x_i}}^+\left[\mathrm{u}\left(\mathrm{x_i}\right)\right] = \frac{1}{\Delta\mathrm{x_i}}\sum_{\mathrm{l=1}}^{\mathrm{L}}\mathrm{a_l}\left\{\mathrm{u}\left(\mathrm{x_i} + \mathrm{l}\Delta\mathrm{x_i}\right) - \mathrm{u}\left[\mathrm{x_i} - (\mathrm{l}-1)\Delta\mathrm{x_i}\right]\right\},
$$

$$
\mathrm{L}_{\mathrm{x_i}}^-\left[\mathrm{u}\left(\mathrm{x_i}\right)\right] = \frac{1}{\Delta\mathrm{x_i}}\sum_{\mathrm{l=1}}^{\mathrm{L}}\mathrm{a_l}\left\{\mathrm{u}\left(\mathrm{x_i} + (\mathrm{l}-1)\Delta\mathrm{x_i}\right) - \mathrm{u}\left[\mathrm{x_i} - \mathrm{l}\Delta\mathrm{x_i}\right]\right\}.
$$

The schemes of other components can also be derived according to the same method.