Numerical modelling of viscoelastic waves by a pseudospectral domain decomposition method

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A first approach may be to take combinations of Taylor series at neighbouring points

\[
\begin{align*}
    f(x_{i+1}) &= f(x_i) + (\Delta x)f'(x_i) + \frac{(\Delta x)^2}{2}f''(x_i) + O((\Delta x)^3), \\
    f(x_{i-1}) &= f(x_i) - (\Delta x)f'(x_i) + \frac{(\Delta x)^2}{2}f''(x_i) + O((\Delta x)^3).
\end{align*}
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If \( (x_{i+1} - x_i) = (x_i - x_{i-1}) = \Delta x \) then the finite-difference approximations for \( f'(x_i) \) are

\[
\begin{align*}
  f'(x_i) &= \frac{f(x_{i+1}) - f(x_i)}{\Delta x} + O(\Delta x), \\
  f'(x_i) &= \frac{f(x_i) - f(x_{i-1})}{\Delta x} + O(\Delta x), \\
  f'(x_i) &= \frac{f(x_{i+1}) - f(x_{i-1})}{2\Delta x} + O(\Delta x^2).
\end{align*}
\]
A more general approach is to build a **Lagrange interpolating polynomial** and differentiate that.

![Lagrange Polynomials](image)

Figure: 3 node Lagrange polynomials defined on equally spaced nodes and the resulting interpolation.
The 3 point differentiation matrix that act on the sampled values of $f$ and returns approximately the sampled values of $f'$ is

\[
\frac{1}{2\Delta x} \begin{pmatrix}
-3 & 4 & -1 & 0 & 0 & \ldots & 0 \\
-1 & 0 & 1 & 0 & 0 & \ldots & 0 \\
0 & -1 & 0 & 1 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \ldots & 0 & -1 & 0 & 1 & 0 \\
0 & \ldots & \ldots & 0 & -1 & 0 & 1 \\
0 & \ldots & \ldots & 0 & 1 & -4 & 3
\end{pmatrix}
\]
We can generalize this approach until the differentiation matrix is fully populated, but the nodes must be chosen carefully due to Runge’s phenomenon.

Figure: 11 Lagrange polynomials defined on equally spaced nodes.
Two popular choices are the **Chebyshev** and **Legendre** points.

Figure: Chebyshev and Legendre polynomials.
Each set of nodes has an associated *pseudospectral differentiation matrix* $D$ that exactly differentiates the interpolating polynomial.
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And a set of *Gauss-Lobatto* integration weights $w$ that are exact for polynomials of degree less than or equal $2N - 1$, where $N$ is number of points.

$$\int_{-1}^{1} f(x)g(x)dx = \sum_{i=1}^{N} f(x_i)g(x_i)w_i.$$
\[ \|D_N f - f'\|_\infty \text{ for } f = x^{10}. \ f = [f(x_0), ..., f(x_N)]^T. \ D_N \text{ is the } (N + 1) \times (N + 1) \text{ pseudospectral differentiation matrix.} \]

Figure: Convergence to machine-precision
To compare accuracies, consider an ugly function

\[ f(x) = x(1 + \sin(10\pi \exp(-10x^2))) \] and its derivative.

Figure: An ugly function \( f(x) = x(1 + \sin(10\pi \exp(-10x^2))) \) and its derivative.
Error Vs. Number of Points

Figure: Various derivative approximations
Figure: Final number of points, times and errors for Chebyshev and 8th order finite differences.
We are concerned with solving wave equations of the form

$$\rho \ddot{u}_i = \partial_j \sigma_{ij}(u) + f_i, \quad x \in \Omega, \quad t > 0$$
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For a Kelvin-Voigt material

\[ \sigma(u) = E\varepsilon + \eta \dot{\varepsilon} (1 - D) \]

\[ \sigma_{ij}(u) = \lambda \nabla \cdot u \delta_{ij} + 2\mu \varepsilon_{ij}(u) + \lambda' \nabla \cdot v \delta_{ij} + 2\mu' \varepsilon_{ij}(v) (N - D) \]
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\( \lambda \) and \( \mu \) are the elastic parameters, \( \lambda' \) and \( \mu' \) are the anelastic parameters.
Let \( u_j(x, z, t) = \hat{u}_j(x, z)e^{i\omega t} \), then

\[
\sigma_{ij} = \lambda \nabla \cdot \hat{u}\delta_{ij} + 2\mu\varepsilon_{ij}(\hat{u}) + i\omega (\lambda' \nabla \cdot \hat{u}\delta_{ij} + 2\mu'\varepsilon_{ij}(\hat{u})) = \Lambda \nabla \cdot \hat{u}\delta_{ij} + 2\mathcal{M}\varepsilon_{ij}(\hat{u})
\]
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= \Lambda \nabla \cdot \hat{u} \delta_{ij} + 2M \varepsilon_{ij}(\hat{u})
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\( \Lambda = \lambda + i\omega \lambda' \) and \( M = \mu + i\omega \mu' \) are the complex Lamé parameters dependent on the frequency \( \omega \).
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The complex P and S wave velocities are defined as

\[
\hat{V}_p = \sqrt{\frac{\Lambda + 2M}{\rho}}, \quad \text{and} \quad \hat{V}_s = \sqrt{\frac{M}{\rho}}
\]
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$$\hat{V}_p = \sqrt{\frac{\Lambda + 2M}{\rho}}, \quad \text{and} \quad \hat{V}_s = \sqrt{\frac{M}{\rho}}$$

The frequency-dependent P and S wave quality factors

$$Q_p = \frac{\lambda + 2\mu}{\omega(\lambda' + 2\mu')}, \quad Q_s = \frac{\mu}{\omega\mu'}.$$
The elastic parameters, $\lambda$ and $\mu$ are

$$\mu = \rho V_s^2 g(Q_s), \quad \lambda = \rho V_p^2 g(Q_p) - 2\mu$$
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The anelastic parameters are

$$\lambda' = \frac{1}{\omega} \left( \frac{\lambda + 2\mu}{Q_p} - \frac{2\mu}{Q_s} \right) \quad \mu' = \frac{1}{\omega Q_s}.$$
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$$\mu = \rho V_s^2 g(Q_s), \quad \lambda = \rho V_p^2 g(Q_p) - 2\mu$$

The anelastic parameters are

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$g$ is obtained algebraically from the above equations as

$$g(Q) = \frac{1}{2} (1 + Q^{-2})^{-1/2} (1 + (1 + Q^{-2})^{-1/2}).$$
Consider a 2-layer, 2-D model.
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The idea is to solve the wave equation in each subdomain and connect them using interface conditions.
Let’s ignore the force term for now leaving

\[ \rho \ddot{u}_i = \partial_j \sigma_{ij}(\mathbf{u}) \]
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Multiplying both sides by an arbitrary function $\phi$ and integrating over space we obtain

$$\int_{\Omega} \rho \ddot{u}_i \phi d\Omega = \int_{\Omega} \partial_j \sigma_{ij}(\mathbf{u}) \phi d\Omega$$
Let’s ignore the force term for now leaving

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Multiplying both sides by an arbitrary function \( \phi \) and integrating over space we obtain

\[ \int_{\Omega} \rho \dddot{u}_i \phi d\Omega = \int_{\Omega} \partial_j \sigma_{ij}(u) \phi d\Omega \]

Now we split the integral over the two regions

\[ \sum_{k=1}^{2} \int_{\Omega^k} \rho \dddot{u}^k_i \phi d\Omega^k = \sum_{k=1}^{2} \int_{\Omega^k} \partial_j \sigma_{ij}(u^k) \phi d\Omega^k \]

Where \( u^k \) are the displacements in the region \( \Omega^k \).
Integrating the right hand side by parts produces

\[ \sum_{k=1}^{2} \int_{\Omega^k} \partial_j \sigma_{ij}(u^k) \phi d\Omega^k \]

\[ = \sum_{k=1}^{2} \left\{ \oint_{\partial\Omega_k} \sigma_{ij}(u^k) \phi n^k_j dS - \int_{\Omega^k} \sigma_{ij}(u^k) \partial_j \phi d\Omega^k \right\} \]

where \( n^k_j \) is the \( j^{th} \) component of the unit normal vector in the region \( \Omega_k \).
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where \(n_j^k\) is the \(j^{th}\) component of the unit normal vector in the region \(\Omega_k\).

We now have

\[\sum_{k=1}^{2} \int_{\Omega_k} \left\{ \rho i u^k_i \phi + \sigma_{ij}(u^k) \partial_j \phi \right\} d\Omega^k = \sum_{k=1}^{2} \oint_{\partial \Omega_k} \sigma_{ij}(u^k) \phi n_j^k dS\]
At the free-surface the appropriate stresses should disappear (zero-traction).

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- Continuity of displacement

$$u^1|_{\Gamma_B} = u^2|_{\Gamma_B}$$
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   - Continuity of stress
     \[ \sigma_{ij}(u^1) |_{\Gamma_B} = \sigma_{ij}(u^2) |_{\Gamma_B} \]
➤ These conditions determine the reflection and transmission coefficients at the interface.
- Continuity is enforced by construction of the basis elements.

Figure: Interface function in 1-D.
Higher-dimensional constructions use product-bases.

Figure: Interface function in 2-D.
Terms involving stresses are enforced by modifying the surface integrals:

\[ \int_{\partial\Omega_1} \sigma_{ij}(\mathbf{u}^1)n_j^1 \phi dS + \int_{\partial\Omega_2} \sigma_{ij}(\mathbf{u}^2)n_j^2 \phi dS \]

\[ \Gamma_N \quad \sigma_{ij}(\mathbf{u}^1)n_j^1|_{\Gamma_N} = 0 \]

\[ \Gamma_W \quad \sigma_{ij}(\mathbf{u}^1)|_{\Gamma_W} = \sigma_{ij}(\mathbf{u}^2)|_{\Gamma_B} \]

\[ \Gamma_S \quad \Gamma_E \]

\[ \Omega_1 \quad \Omega_2 \]
The discretization results in a system of equations for the $k^{th}$ element

$$M_k^k \ddot{u}_i^k(t) + \sum_j \hat{K}_{ij}^k \dot{u}_i^k(t) + \sum_j K_{ij}^k u_j^k(t) = M_f^k f_i^k(t).$$
The discretization results in a system of equations for the $k^{th}$ element

$$M^k \ddot{u}_i^k(t) + \sum_j \hat{K}_{ij}^k \dot{u}_i^k(t) + \sum_j K_{ij}^k u_j^k(t) = M^k f_i^k(t).$$

Absorbing boundaries are enforced by replacing interior derivatives with one-way wave equations.

$$\partial_z u_1 \leftarrow -\frac{1}{V_s} v_1 + \frac{V_s - V_p}{V_s} \partial_x u_2, \ x \in \Gamma_S$$
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$$
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This is implemented by modifying the damping matrices $\hat{K}^k_{ij}$ in the elements along the absorbing boundaries.
The discretization results in a system of equations for the $k^{th}$ element

$$M^k \dddot{u}_i^k(t) + \sum_{j} \hat{K}_{ij}^k \ddot{u}_i^k(t) + \sum_{j} K_{ij}^k u_j^k(t) = M^k f_i^k(t).$$

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The system is written in block form

$$\begin{pmatrix} M & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} \dot{V} \\ \dot{U} \end{pmatrix} + \begin{pmatrix} \hat{K} & K \\ I & 0 \end{pmatrix} \begin{pmatrix} V \\ U \end{pmatrix} = \begin{pmatrix} F \\ 0 \end{pmatrix}$$
To show the high-frequency damping present in the anelastic part of the model we purposefully choose a grid too coarse to represent the source wavelet (30 Hz Ricker). The boundary is at \( z = 250\text{m} \). The model is time-stepped using a \( 4^{th} \) low-storage explicit Runge-Kutta (LSERK) method.

<table>
<thead>
<tr>
<th>( \Omega )</th>
<th>( \rho )</th>
<th>( V_p )</th>
<th>( V_s )</th>
<th>( Q_p )</th>
<th>( Q_s )</th>
</tr>
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<tbody>
<tr>
<td>( \Omega_1 )</td>
<td>2.06</td>
<td>2400</td>
<td>1500</td>
<td>( \infty )</td>
<td>( \infty )</td>
</tr>
<tr>
<td>( \Omega_2 )</td>
<td>2.06</td>
<td>2400</td>
<td>1500</td>
<td>10</td>
<td>10</td>
</tr>
</tbody>
</table>
t=0.07 sec.
t = 0.15 sec.
t = 0.23 sec.
t = 0.32 sec.
t=0.4 sec.
t=0.48 sec.
Consider the case of a reflection strictly from a difference in $Q_p$ and $Q_s$. The boundary is at $z = 500m$ and is again time-stepped using 4th order LSERK.

<table>
<thead>
<tr>
<th></th>
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<td>1500</td>
<td>20</td>
<td>30</td>
</tr>
</tbody>
</table>
Thank you
Thank you
(a) Original trace.

(b) Clipped trace.
Figure: Horizontal displacement.
Figure: Vertical Displacement.
Figure: Horizontal velocity.
Figure: Vertical velocity.
Thank you!

- Chris Bird
- Michael Lamoureux
- Crewes
- Potsi
- mprime
- Pims
- Nserc