

Quantifying parameter resolution for elastic and anisotropic FWI with multi-parameter Hessian probing

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Objective

- ◇ Full-waveform inversion (FWI) methods are heading for recovering elastic and anisotropic properties for reservoir characterization;
- ◇ Complex physics of subsurface medium makes the inverse problems much more challenging;
- ◇ Parameter resolution issue arising from inverting multiple physical parameters;
- ◇ Quantifying the resolving abilities of different parameterizations lies at the heart of multi-parameter FWI;
- ◇ Most of traditional parameter resolution studies are based on "scattering" or "radiation" patterns following Tarantola (1986);
- ◇ Evaluate the parameter resolution via probing the multi-parameter Hessian;

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- **Review the basic theory of full-waveform inversion (FWI)**
- **Parameter resolution issue in multi-parameter FWI**
- **Quantifying parameter resolution via probing multi-parameter Hessian**
- **Numerical Examples**
- **Conclusions**

1. Review basic theory of full-waveform inversion (FWI)

Basic Theory of FWI

Search direction $\Delta \mathbf{m}$ is the solution of the Newton system:

$$\mathbf{H}_k \Delta \mathbf{m}_k = -\mathbf{g}_k \quad \rightarrow \quad \Delta \mathbf{m}_k = -\mathbf{H}_k^{-1} \mathbf{g}_k$$

Hessian matrix is an extremely large and dense matrix.

It is unaffordable to calculate, store and invert it explicitly for large-scale inverse problem.

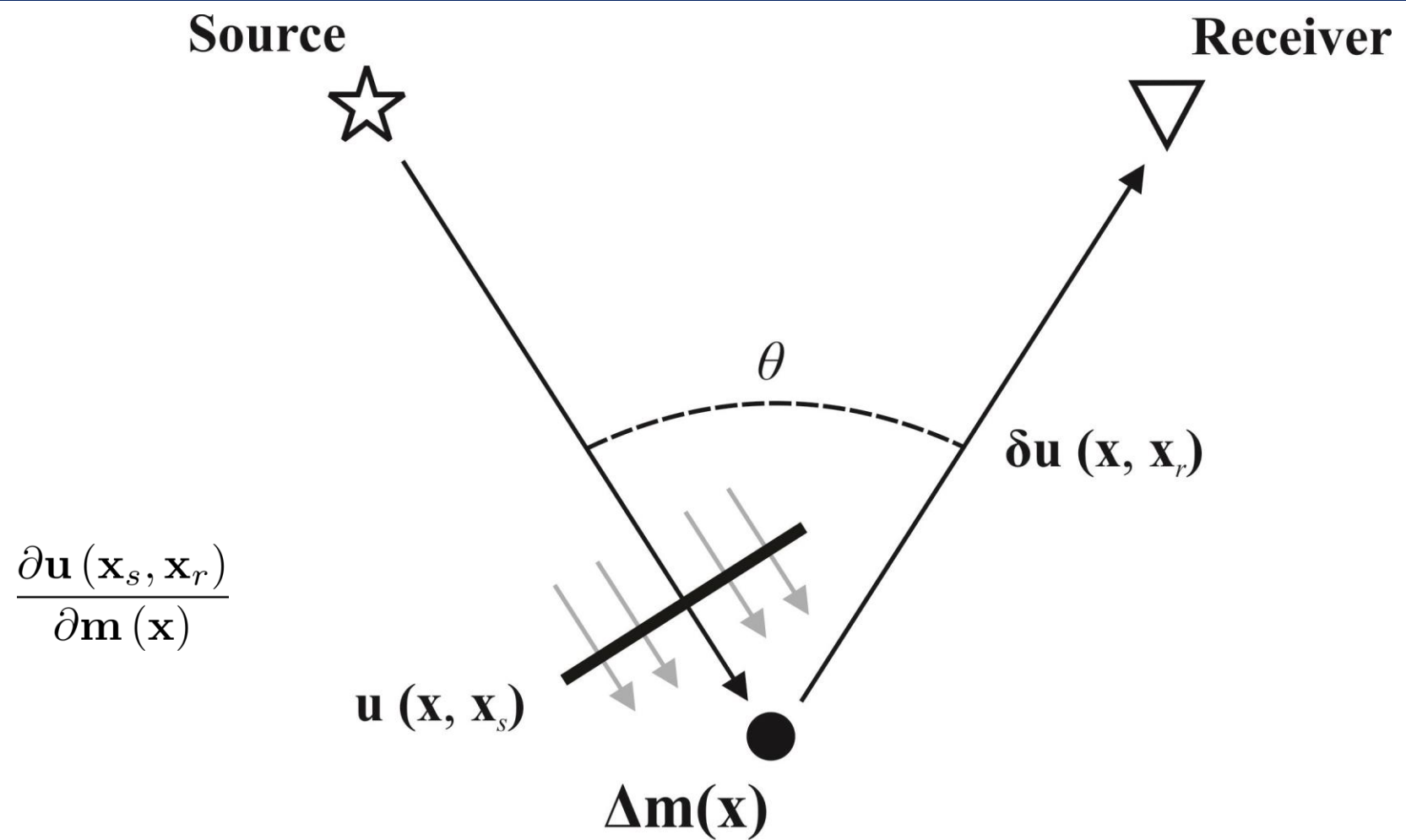
Different optimization methods have different approximations of the inverse Hessian \mathbf{H}^{-1} .

The matrix multiplication can be written as an integral formulation:

$$\mathbf{g} = - \int \int H(\mathbf{x}, \mathbf{x}') \Delta m(\mathbf{x}') d\mathbf{x}' d\mathbf{x}$$

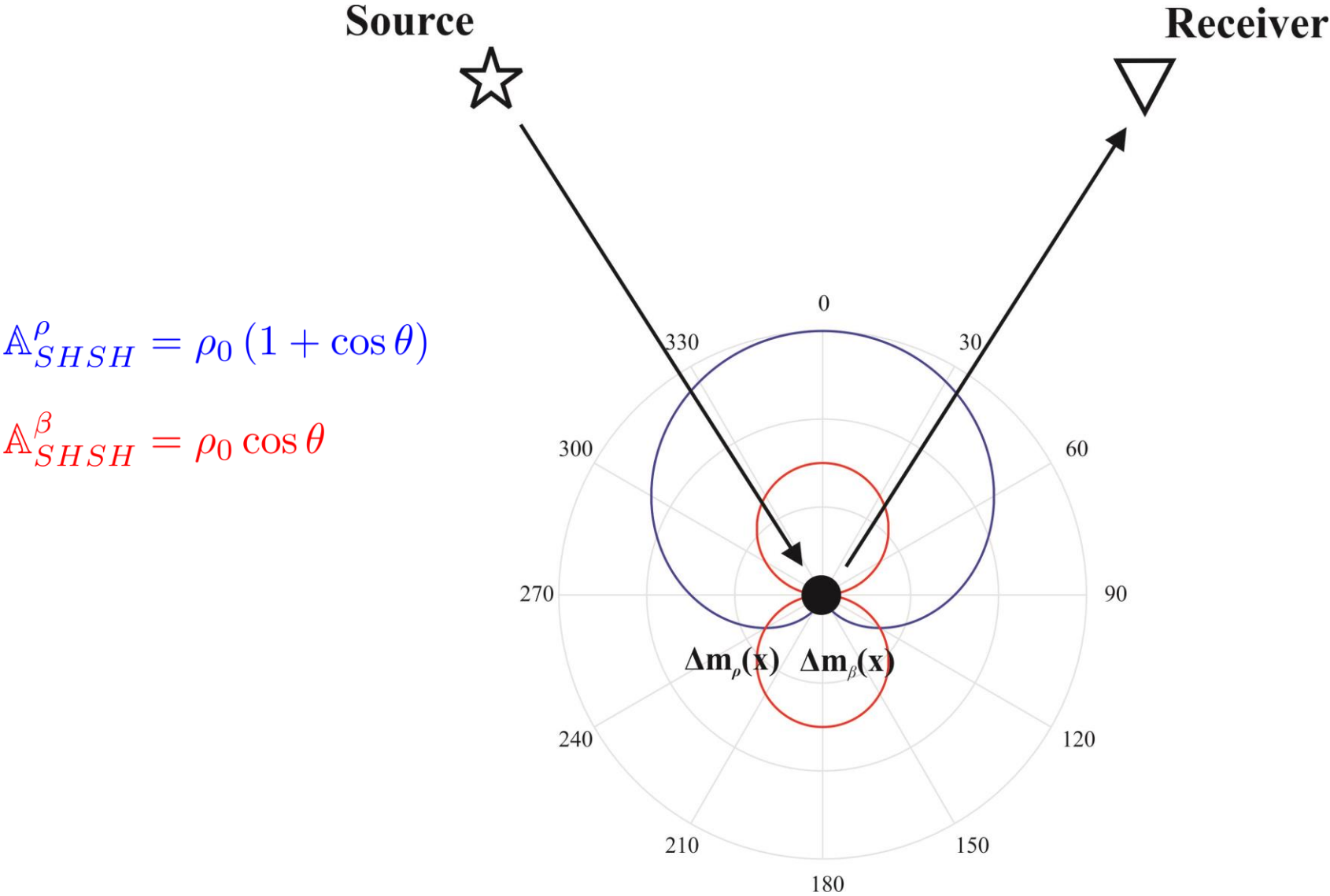
2. Parameter Resolution Issue in Multi-parameter FWI

Parameter resolution analysis based on scattering patterns



Isotropic and Homogeneous

Parameter resolution analysis based on scattering patterns



Parameter resolution analysis based on scattering patterns

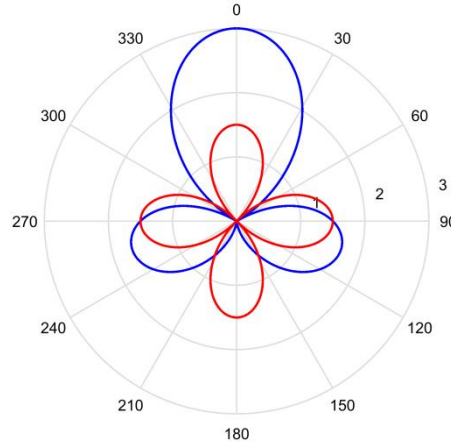
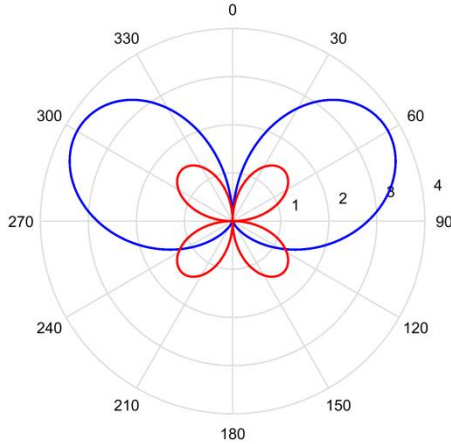
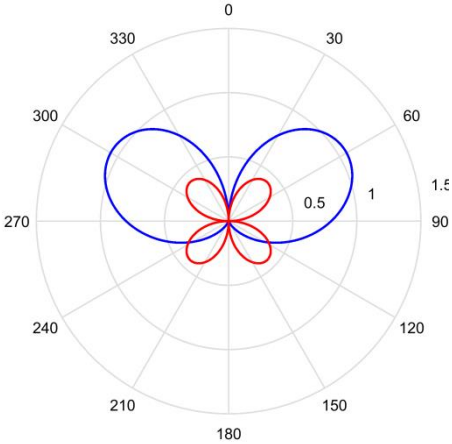
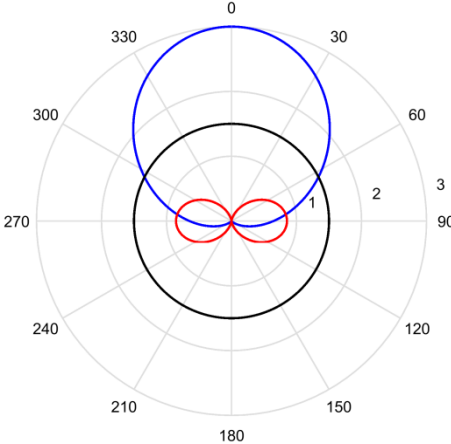
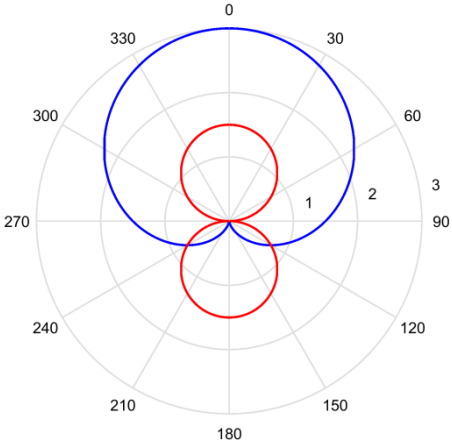
SH → SH

P → P

P → SV

SV → P

SV → SV



$$A_{SHSH}^\rho$$

$$A_{SHSH}^\beta$$

$$A_{PP}^\rho$$

$$A_{PP}^\beta$$

$$A_{PP}^\alpha$$

$$A_{PSV}^\rho$$

$$A_{PSV}^\beta$$

$$A_{SVP}^\rho$$

$$A_{SVP}^\beta$$

$$A_{SVSV}^\rho$$

$$A_{SVSV}^\beta$$

Parameter resolution analysis based on scattering patterns

Disadvantages

◇ Isotropic and homogeneous background assumption:

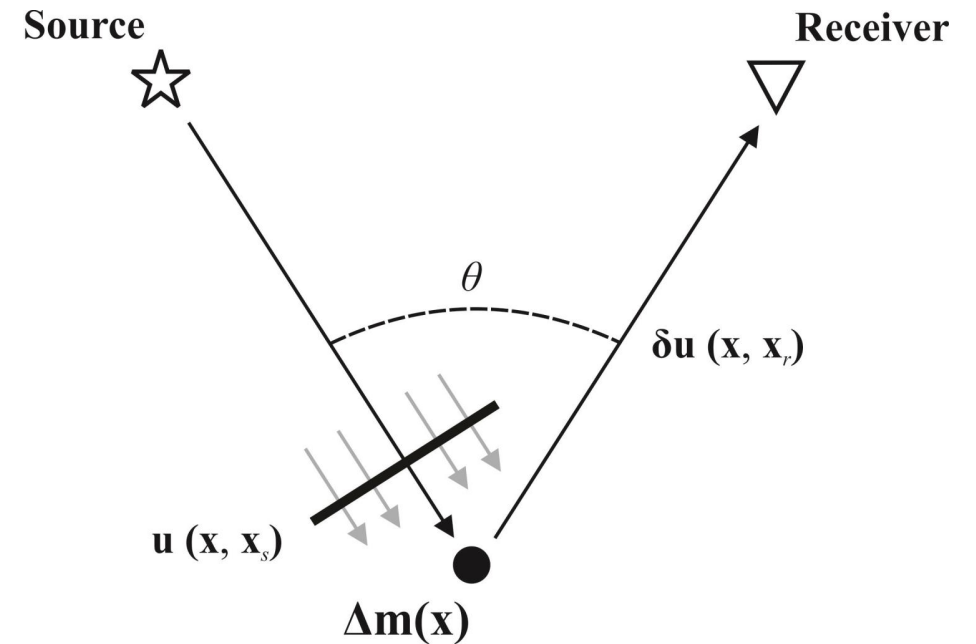
- Ignore more complex model (anisotropy);
- Ignore complex geological structure;

◇ Incident plane wave with far-field assumption:

- Ignore geometrical spreading;
- Ignore travel time contribution;

◇ Overlapping the scattering patterns for analysis:

- Only consider amplitudes and ignore the contributions of phase;
- Only consider the parameter trade-offs at the same location;
- Ignore the parameter trade-offs at adjacent locations;
- Equivalent to the diagonal elements of off-diagonal blocks in multi-parameter Gauss-Newton Hessian;



Isotropic and Homogeneous

3. Quantify the parameter resolution via probing multi-parameter Hessian

Parameter Crosstalk Difficulty

For a multi-parameter inverse problem, the Newton equation becomes:

$$\begin{bmatrix} \mathbf{H}_{\alpha\alpha} & \mathbf{H}_{\alpha\beta} & \mathbf{H}_{\alpha\rho} \\ \mathbf{H}_{\beta\alpha} & \mathbf{H}_{\beta\beta} & \mathbf{H}_{\beta\rho} \\ \mathbf{H}_{\rho\alpha} & \mathbf{H}_{\rho\beta} & \mathbf{H}_{\rho\rho} \end{bmatrix} \begin{bmatrix} \Delta\mathbf{m}_{\alpha} \\ \Delta\mathbf{m}_{\beta} \\ \Delta\mathbf{m}_{\rho} \end{bmatrix} = - \begin{bmatrix} \mathbf{g}_{\alpha} \\ \mathbf{g}_{\beta} \\ \mathbf{g}_{\rho} \end{bmatrix}$$

α is P-wave velocity, β is S-wave velocity, ρ is density

Multi-parameter Hessian has a block structure with 3 diagonal blocks and 6 off-diagonal blocks.

Hessian (or Resolution) matrix is extremely large and difficult to be constructed explicitly.

Physical Interpretations of Multi-parameter Hessian

◇ The elements of multi-parameter Hessian are classified into 4 types;

1. Diagonal elements of the diagonal blocks: $H_{\alpha\alpha}(\mathbf{x}_1, \mathbf{x}_1)$
2. Off-diagonal elements of the diagonal blocks: $H_{\alpha\alpha}(\mathbf{x}_1, \mathbf{x}_M)$
3. Diagonal elements of the off-diagonal blocks: $H_{\beta\alpha}(\mathbf{x}_1, \mathbf{x}_1)$
4. Off-diagonal elements of the off-diagonal blocks: $H_{\beta\alpha}(\mathbf{x}_1, \mathbf{x}_M)$

◇ Diagonal blocks measure spatial resolution;

◇ Off-diagonal blocks measure spatial plus parameter resolution;

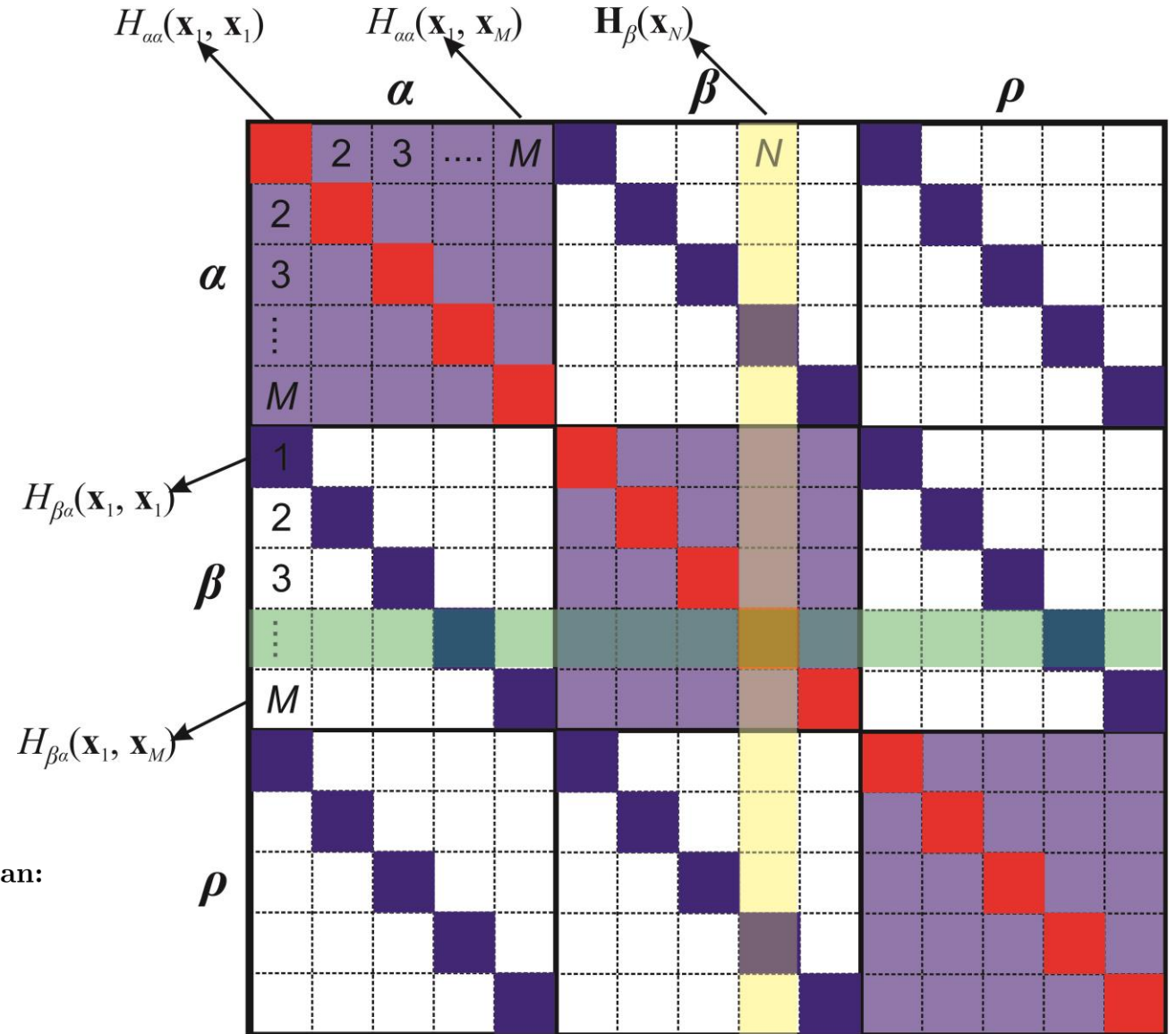
◇ Columns and rows have different physical interpretations:

◇ Rows are blurring kernels;

◇ Columns are multi-parameter point-spread functions;

◇ Overlapping the scattering patterns is partial multi-parameter Hessian:

$$H_{\beta\alpha}^{GN}(\mathbf{x}_1, \mathbf{x}_1) = \frac{\partial \mathbf{u}^\dagger(\mathbf{x}_s, \mathbf{x}_r)}{\partial \mathbf{m}_\beta(\mathbf{x}_1)} \frac{\partial \mathbf{u}^*(\mathbf{x}_s, \mathbf{x}_r)}{\partial \mathbf{m}_\beta(\mathbf{x}_1)}$$



Parameter Crosstalk Difficulty

Newton equation system can be written in an integral formulation:

$$\mathbf{g}_\alpha = - \int \int H_{\alpha\alpha}(\mathbf{x}, \mathbf{x}') \Delta m_\alpha(\mathbf{x}') d\mathbf{x}' d\mathbf{x} - \int \int H_{\alpha\beta}(\mathbf{x}, \mathbf{x}') \Delta m_\beta(\mathbf{x}') d\mathbf{x}' d\mathbf{x} - \int \int H_{\alpha\rho}(\mathbf{x}, \mathbf{x}') \Delta m_\rho(\mathbf{x}') d\mathbf{x}' d\mathbf{x}$$

- ◇ Model perturbations of S-wave velocity and density are mapped into gradient vector of P-wave velocity;
- ◇ Inter-parameter mappings are described by the off-diagonal blocks of multi-parameter Hessian;

Vice versa, we have:

$$\mathbf{g}_\beta = - \int \int H_{\beta\alpha}(\mathbf{x}, \mathbf{x}') \Delta m_\alpha(\mathbf{x}') d\mathbf{x}' d\mathbf{x} - \int \int H_{\beta\beta}(\mathbf{x}, \mathbf{x}') \Delta m_\beta(\mathbf{x}') d\mathbf{x}' d\mathbf{x} - \int \int H_{\beta\rho}(\mathbf{x}, \mathbf{x}') \Delta m_\rho(\mathbf{x}') d\mathbf{x}' d\mathbf{x}$$

$$\mathbf{g}_\rho = - \int \int H_{\rho\alpha}(\mathbf{x}, \mathbf{x}') \Delta m_\alpha(\mathbf{x}') d\mathbf{x}' d\mathbf{x} - \int \int H_{\rho\beta}(\mathbf{x}, \mathbf{x}') \Delta m_\beta(\mathbf{x}') d\mathbf{x}' d\mathbf{x} - \int \int H_{\rho\rho}(\mathbf{x}, \mathbf{x}') \Delta m_\rho(\mathbf{x}') d\mathbf{x}' d\mathbf{x}$$

- ◇ The strength, phase, and spread width of the inter-parameter mappings differ significantly;

Spike Probing of Multi-parameter Hessian

Considering a point perturbation of P-wave velocity $\Delta m_\alpha(\tilde{\mathbf{x}})$:

$$\Delta m_\alpha = \delta(\mathbf{x} - \tilde{\mathbf{x}}), \quad \Delta m_\beta = 0, \quad \Delta m_\rho = 0$$

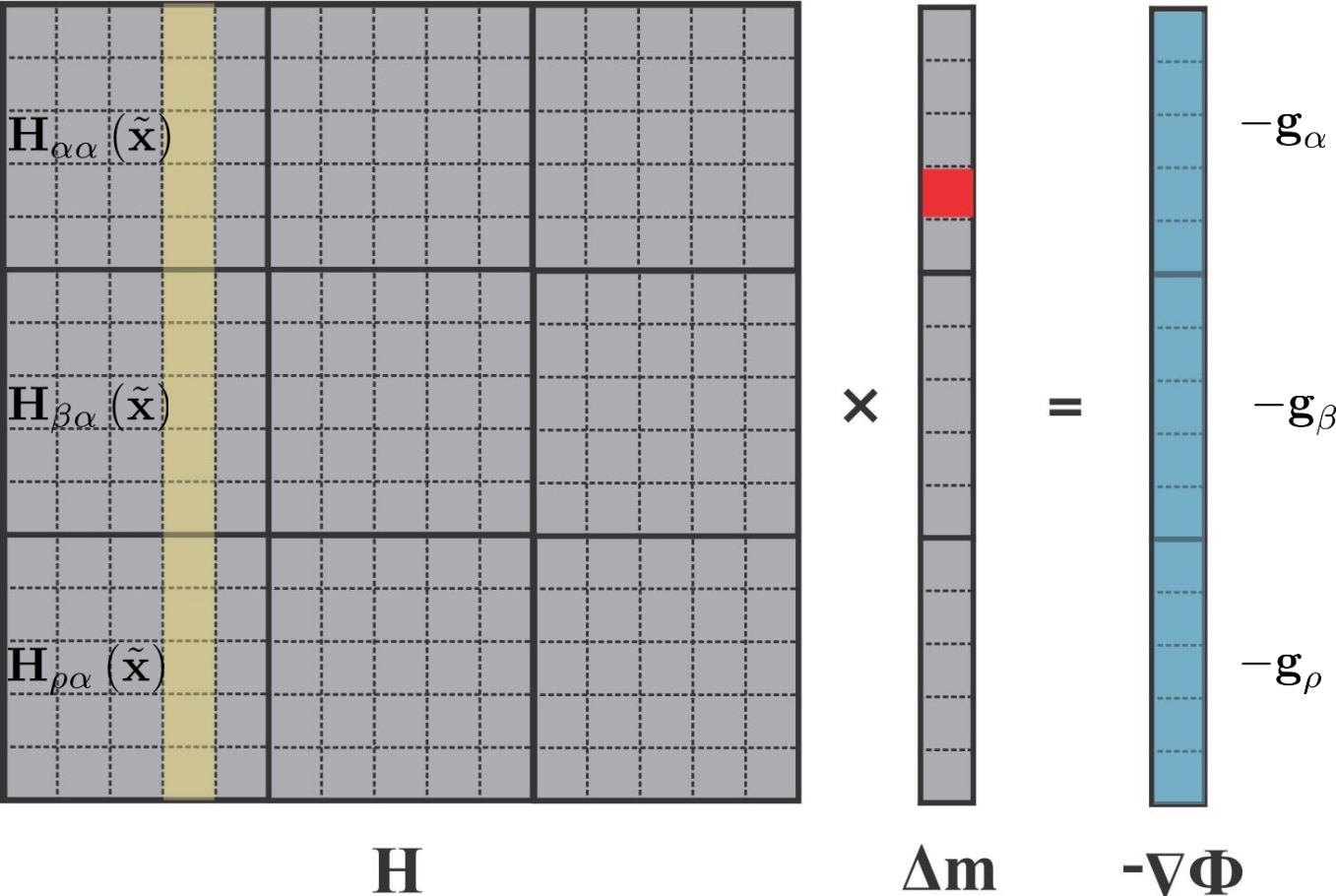
$$\begin{aligned} \mathbf{g}_\alpha &= - \int \int H_{\alpha\alpha}(\mathbf{x}, \mathbf{x}') \Delta m_\alpha(\mathbf{x}') d\mathbf{x}' d\mathbf{x} - \int \int H_{\alpha\beta}(\mathbf{x}, \mathbf{x}') \Delta m_\beta(\mathbf{x}') d\mathbf{x}' d\mathbf{x} - \int \int H_{\alpha\rho}(\mathbf{x}, \mathbf{x}') \Delta m_\rho(\mathbf{x}') d\mathbf{x}' d\mathbf{x} \\ &= - \int \int H_{\alpha\alpha}(\mathbf{x}, \mathbf{x}') \delta(\mathbf{x}' - \tilde{\mathbf{x}}) d\mathbf{x}' d\mathbf{x} - 0 - 0 \\ &= -\mathbf{H}_{\alpha\alpha}(\tilde{\mathbf{x}}) \end{aligned}$$

$\mathbf{g}_\beta = -\mathbf{H}_{\beta\alpha}(\tilde{\mathbf{x}})$ The coupling of P-wave velocity at position $\tilde{\mathbf{x}}$ with S-wave velocity at adjacent positions.

$\mathbf{g}_\rho = -\mathbf{H}_{\rho\alpha}(\tilde{\mathbf{x}})$ The coupling of P-wave velocity at position $\tilde{\mathbf{x}}$ with density at adjacent positions.

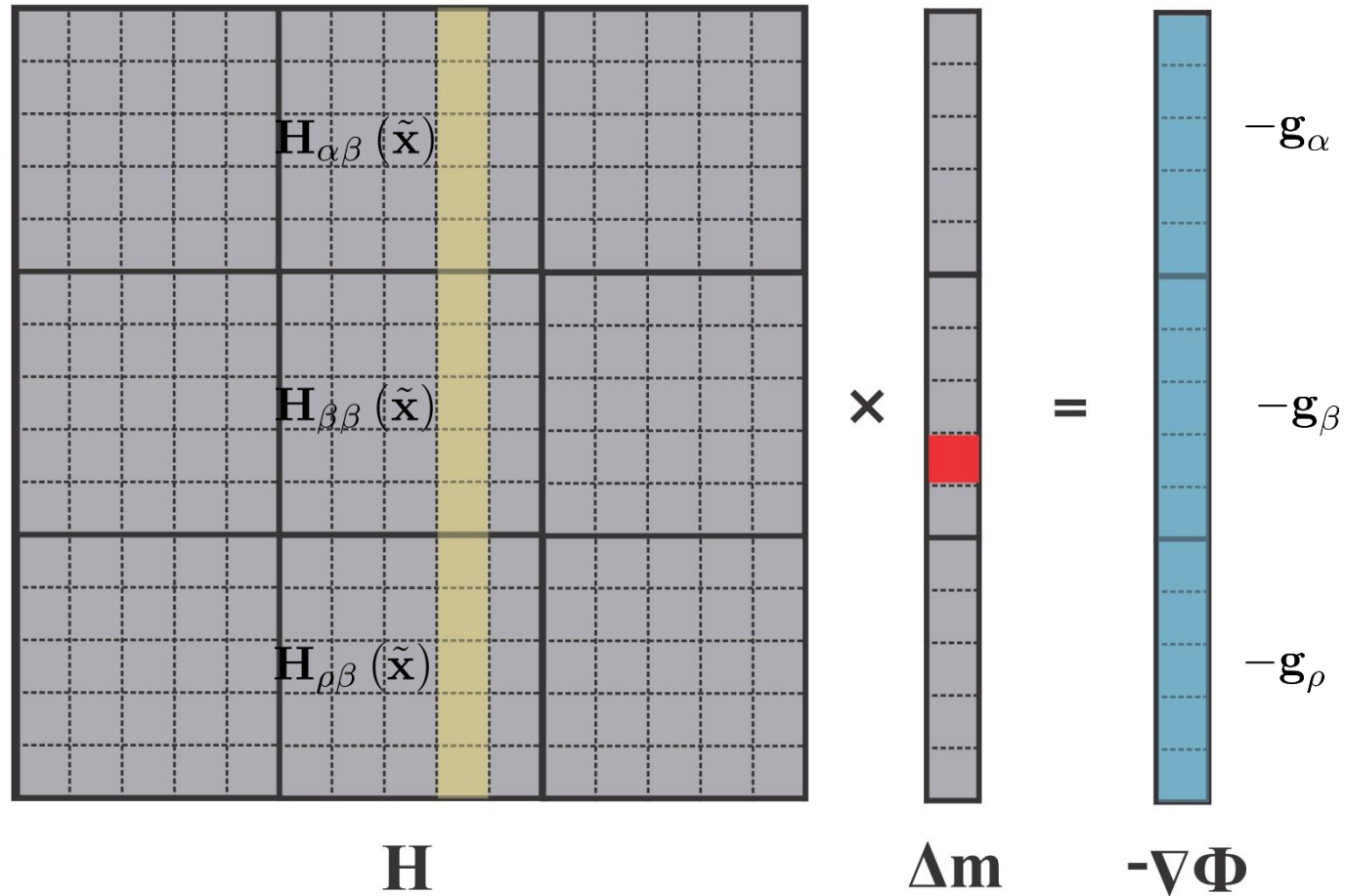
Spike Probing of Multi-parameter Hessian

$$\mathbf{g}_\alpha = -\mathbf{H}_{\alpha\alpha}(\tilde{\mathbf{x}}), \quad \mathbf{g}_\beta = -\mathbf{H}_{\beta\alpha}(\tilde{\mathbf{x}}), \quad \mathbf{g}_\rho = -\mathbf{H}_{\rho\alpha}(\tilde{\mathbf{x}})$$



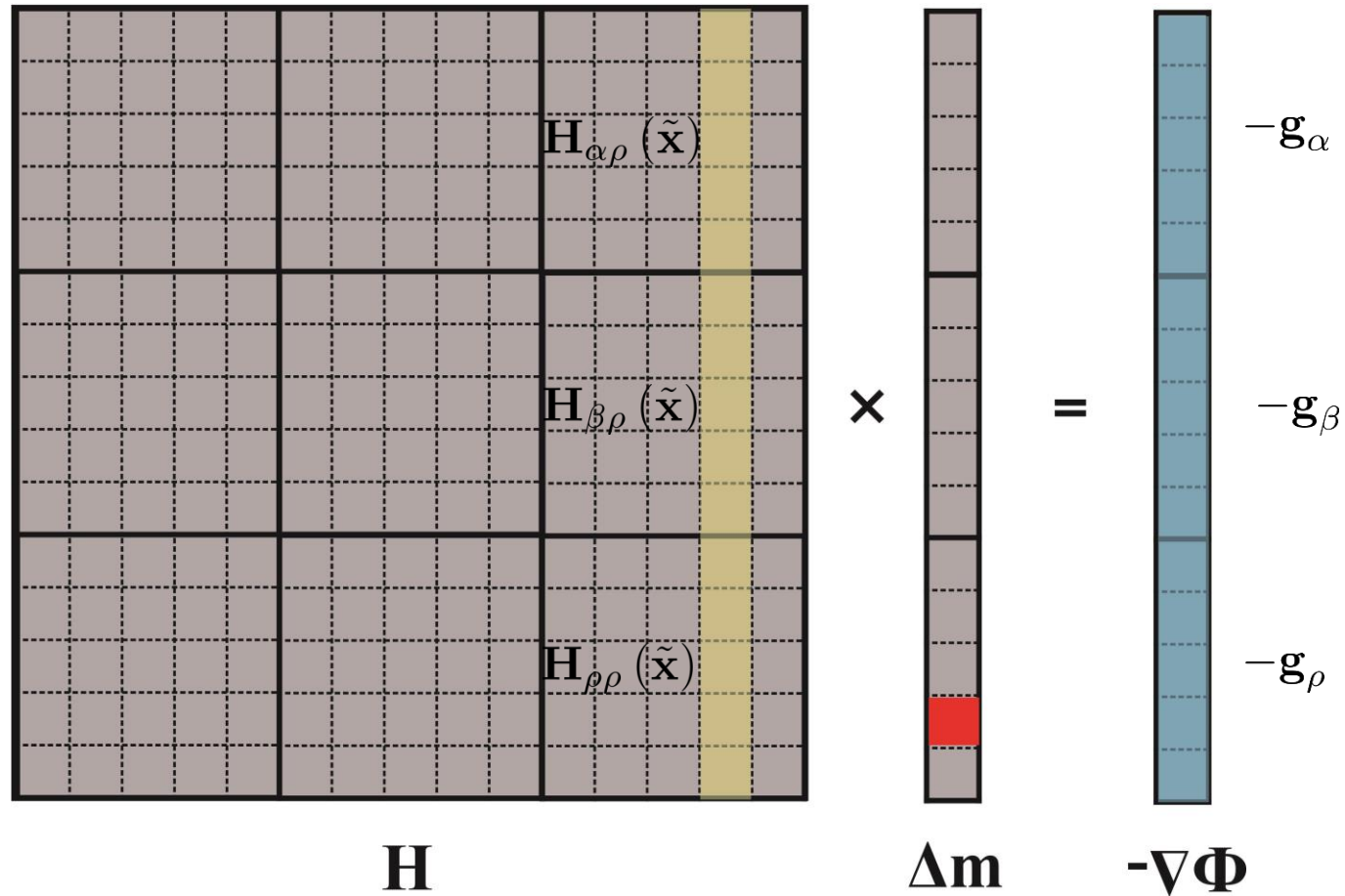
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Spike Probing of Multi-parameter Hessian

$$\mathbf{g}_\alpha = -\mathbf{H}_{\alpha\rho}(\tilde{\mathbf{x}}), \quad \mathbf{g}_\beta = -\mathbf{H}_{\beta\rho}(\tilde{\mathbf{x}}), \quad \mathbf{g}_\rho = -\mathbf{H}_{\rho\rho}(\tilde{\mathbf{x}})$$



Stochastic Probing of Multi-parameter Hessian

Random vector: $\Delta\mathbf{m}_r$

Hessian-vector product: $\mathbf{H}\Delta\mathbf{m}_r$

Compute the product of the Hessian-vector with the random vector:

$$\Delta\mathbf{m}_r^\dagger \mathbf{H} \Delta\mathbf{m}_r$$

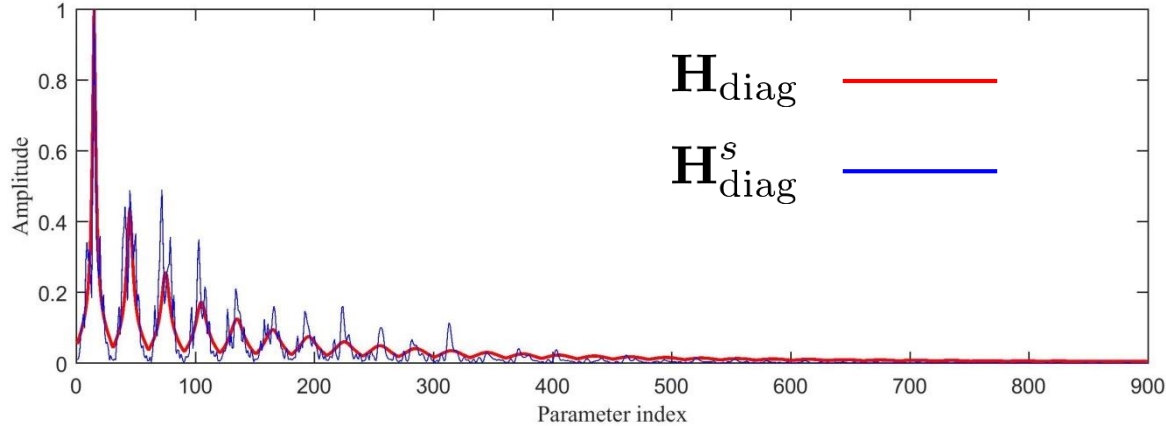
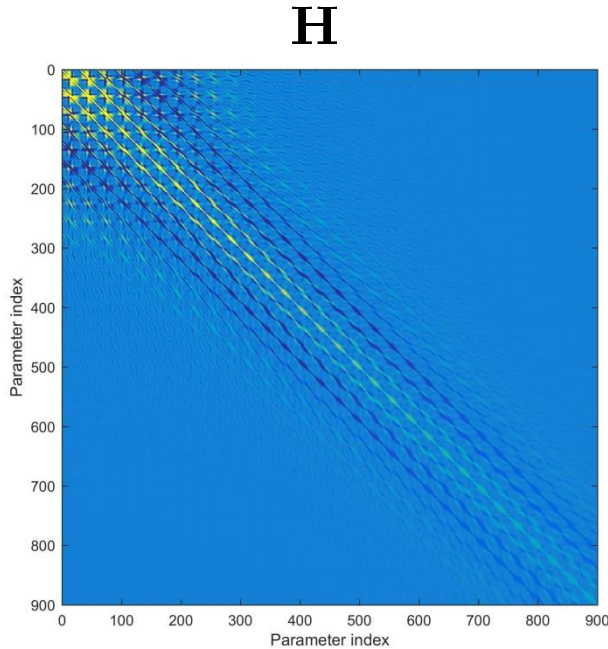
Apply expectation operator with a series of random vector realizations:

$$\sum_{r=1}^{NR} \mathbb{E} (\Delta\mathbf{m}_r^\dagger \mathbf{H} \Delta\mathbf{m}_r) / NR \approx \mathbf{H}_{\text{diag}}$$

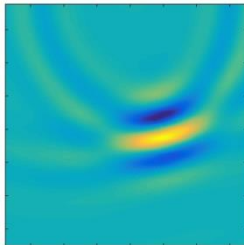
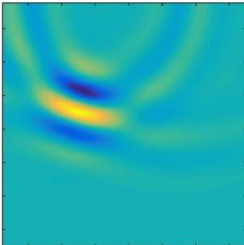
With different random vectors ensemble:

$$\sum_{r=1}^{NR} \mathbb{E} (\Delta\tilde{\mathbf{m}}_r^\dagger \mathbf{H} \Delta\mathbf{m}_r) / NR \approx \mathbf{H}_{\text{diag}}^{\rho\beta}$$

Stochastic Probing of Multi-parameter Hessian

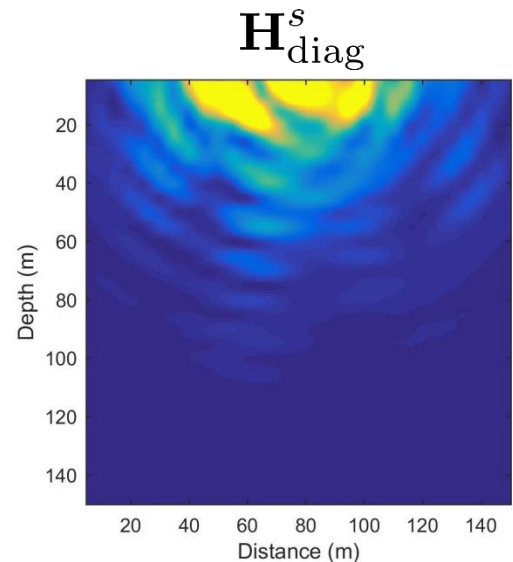
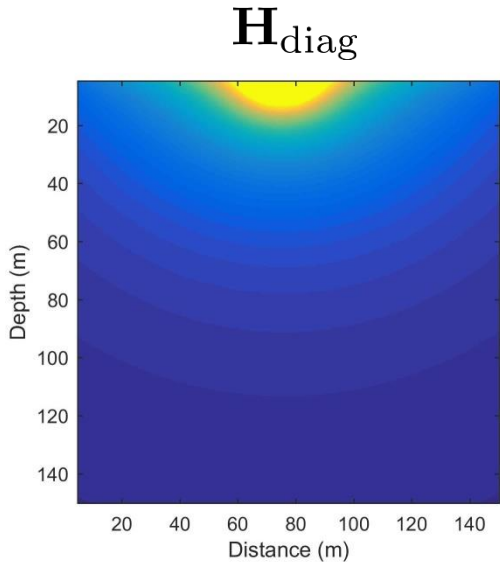


$$\sum_{r=1}^{NR} \mathbb{E} (\Delta \mathbf{m}_r^\dagger \mathbf{H} \Delta \mathbf{m}_r) / NR \approx \mathbf{H}_{\text{diag}}$$



$\mathbf{H}(\mathbf{x}_1)$

$\mathbf{H}(\mathbf{x}_2)$



Benefits of Probing Multi-parameter Hessian

Spike probing method:

- ◇ finite-frequency effects.
- ◇ Evaluate the strength, polarity and spread width of the inter-parameter mappings.
- ◇ Evaluate the coupling effects of different physical parameters at different locations.
- ◇ geometrical spreading, complex model and complex geological structure.
- ◇ local limitation.

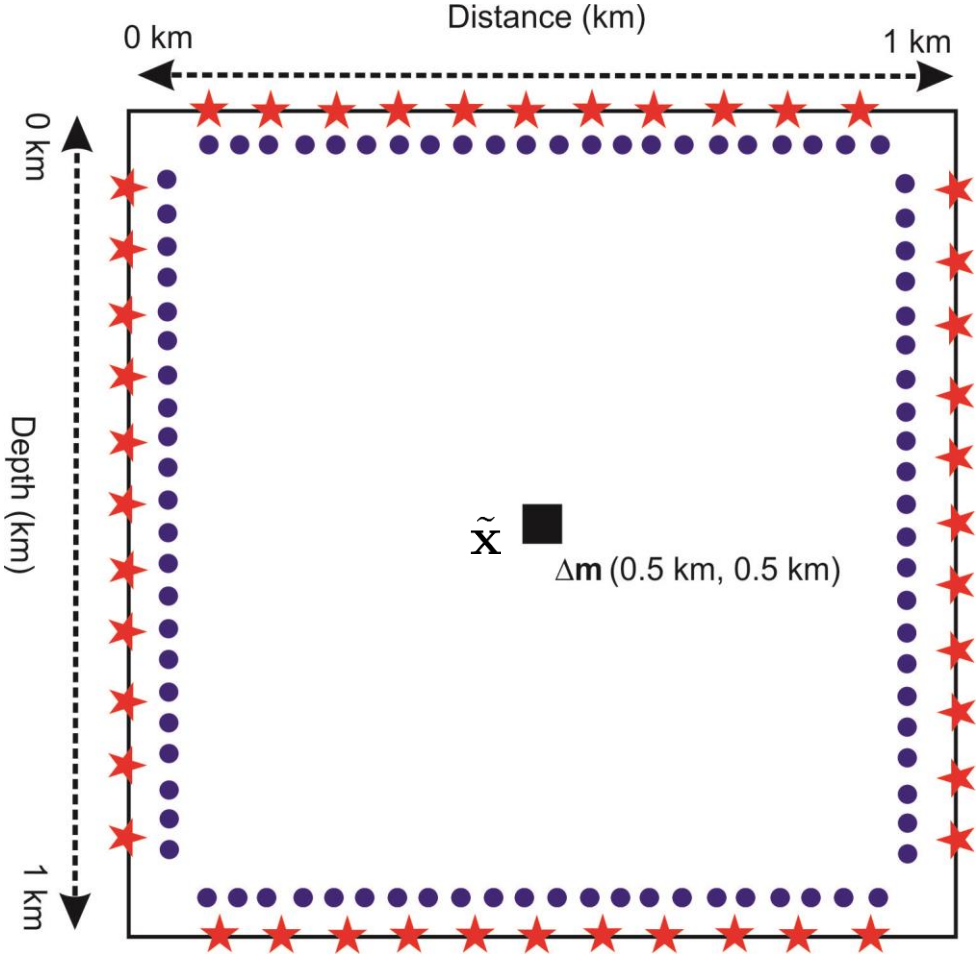
Stochastic probing method:

- ◇ travel time.
- ◇ whole volume.

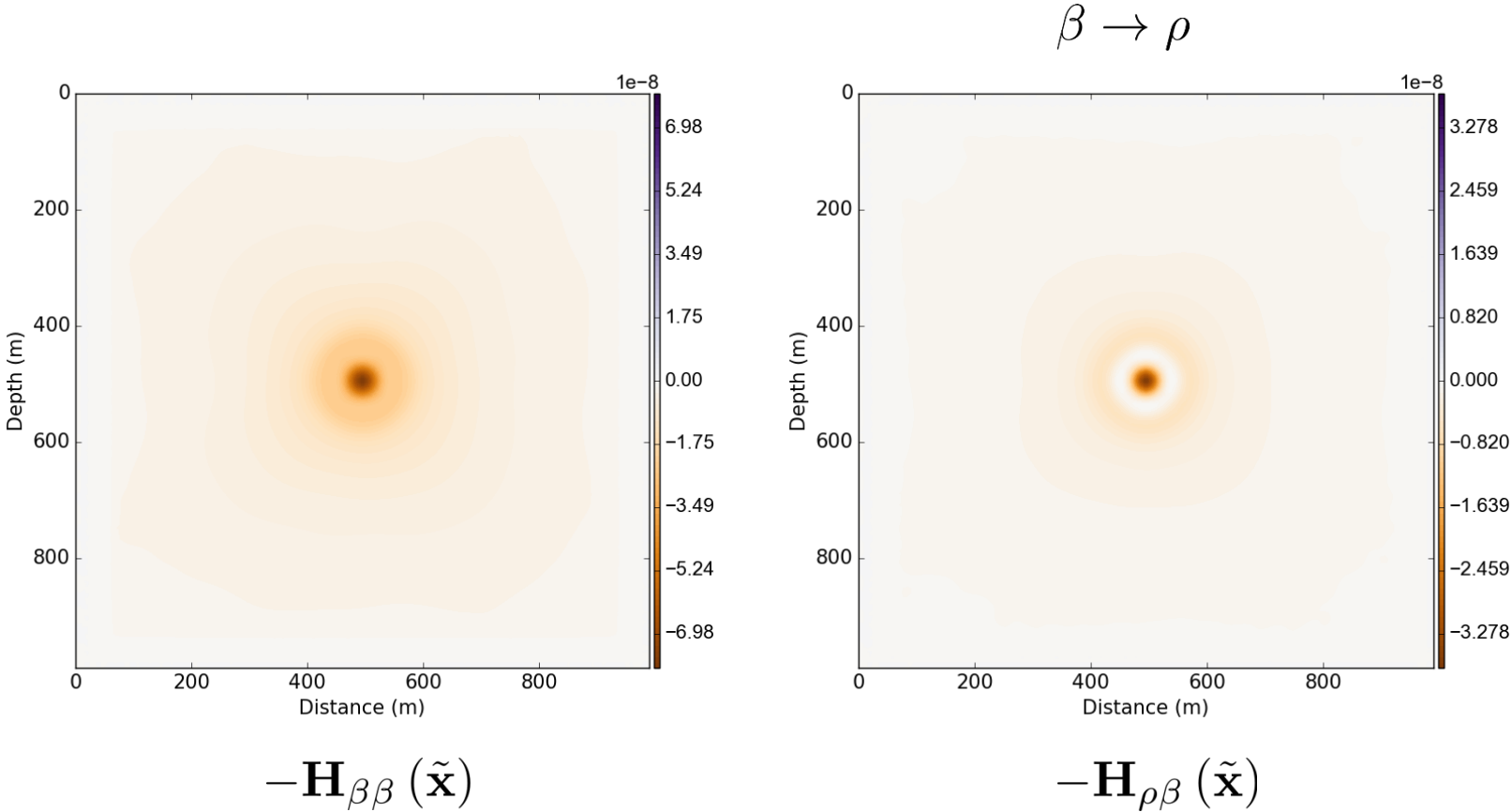
4. Numerical Examples

4.1 Isotropic and Elastic Examples

2D Elastic Example with Perfect Acquisition Geometry

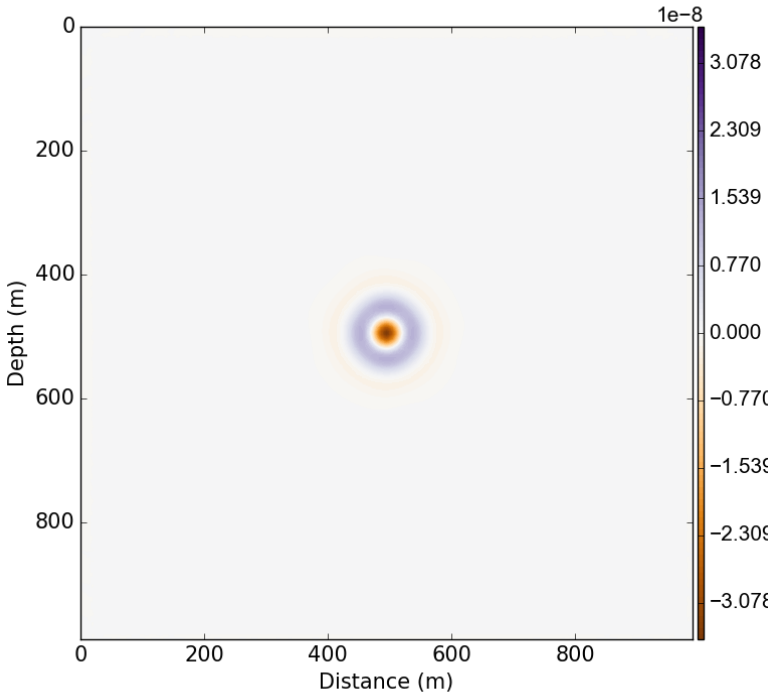


Multi-parameter Point Spread Function: SH Example

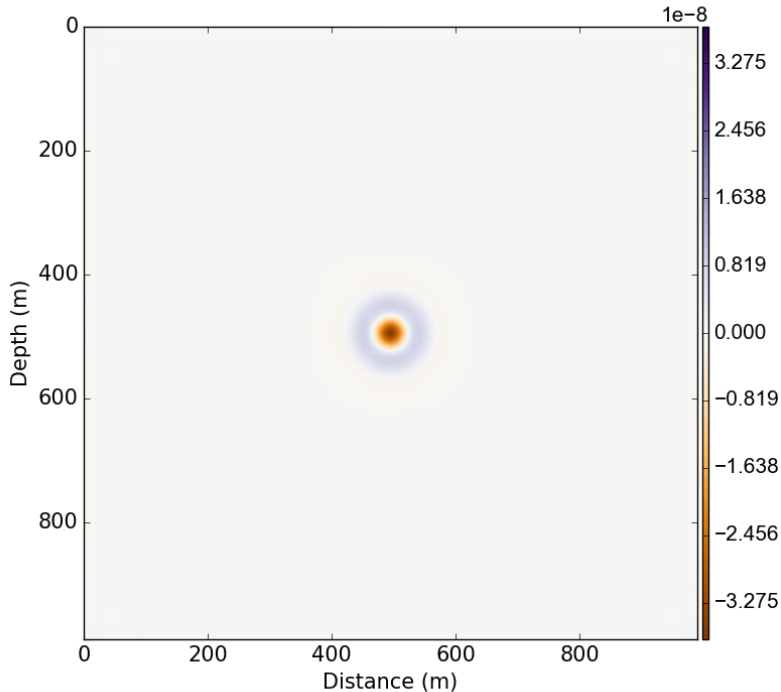


Multi-parameter Point Spread Function: SH Example

$$\rho \rightarrow \beta$$

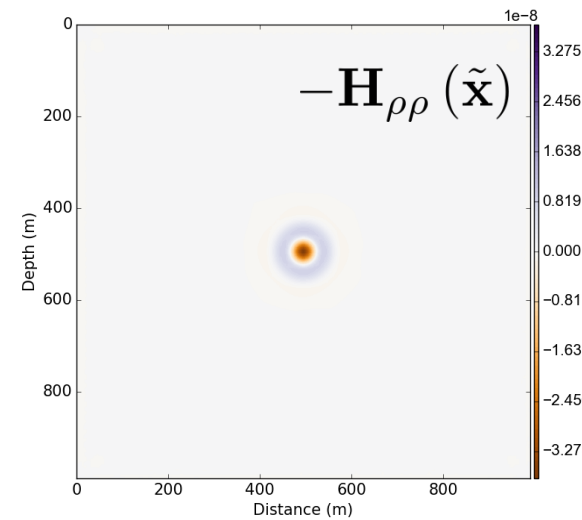
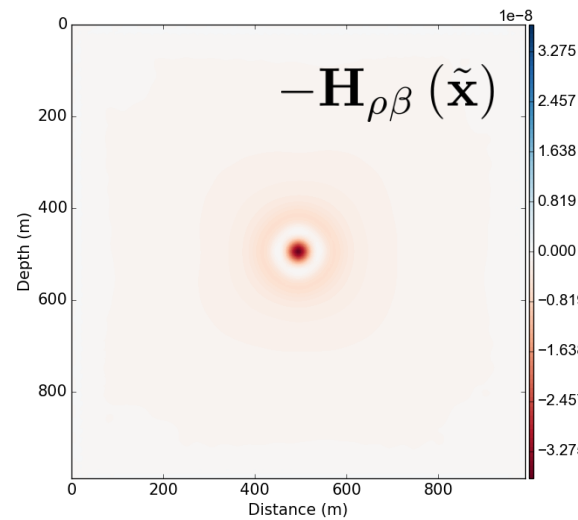
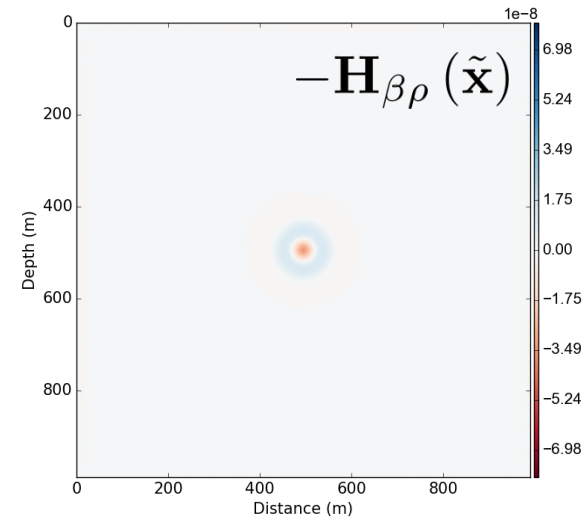
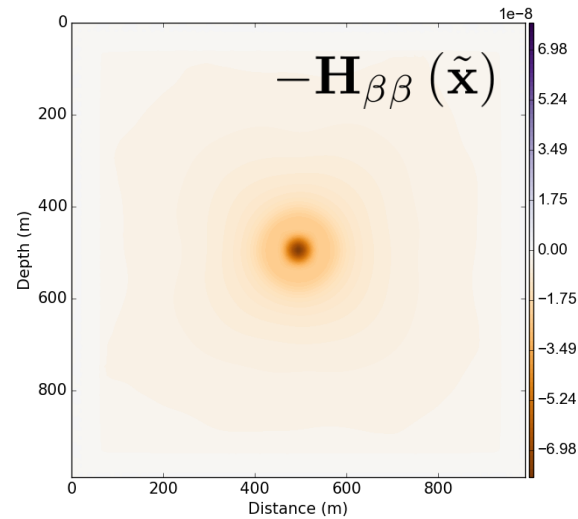


$$-\mathbf{H}_{\beta\rho}(\tilde{\mathbf{x}})$$

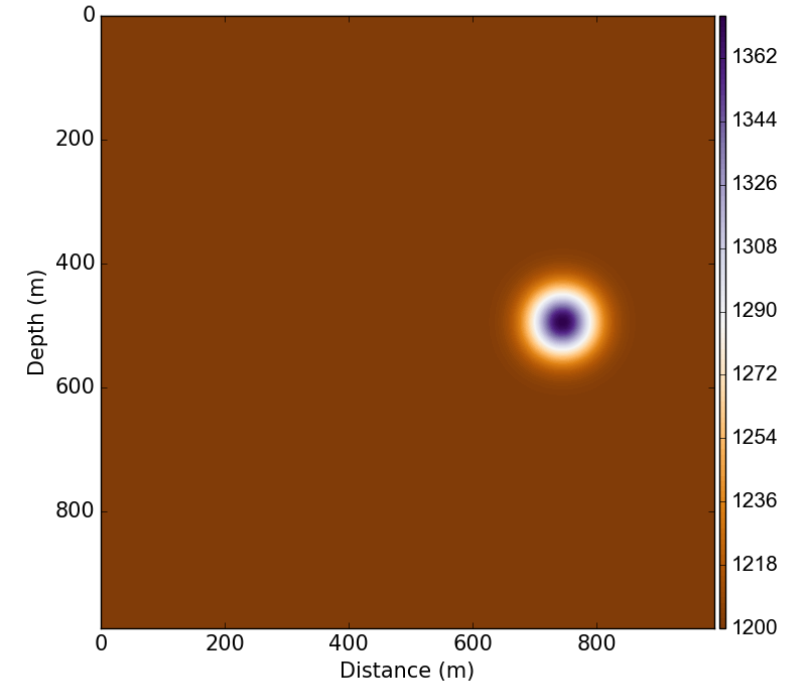
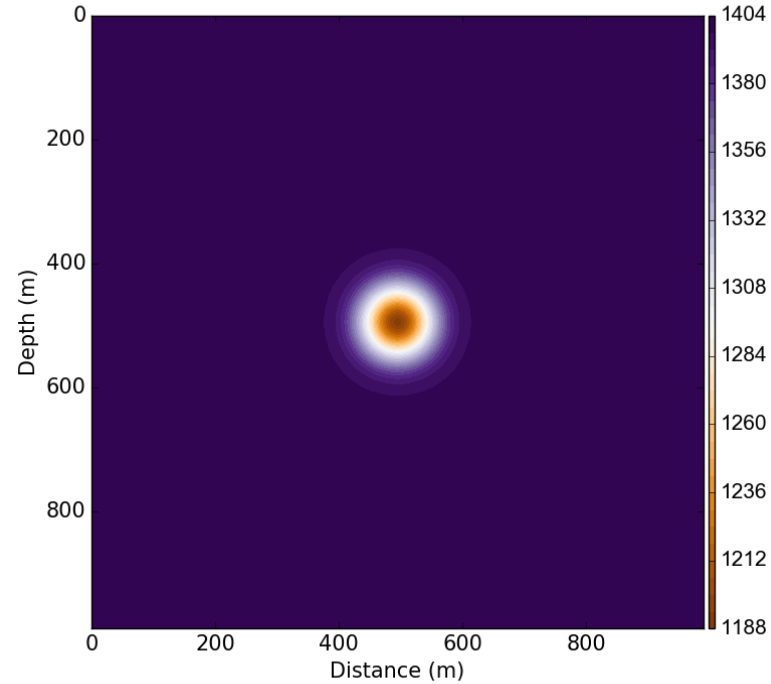
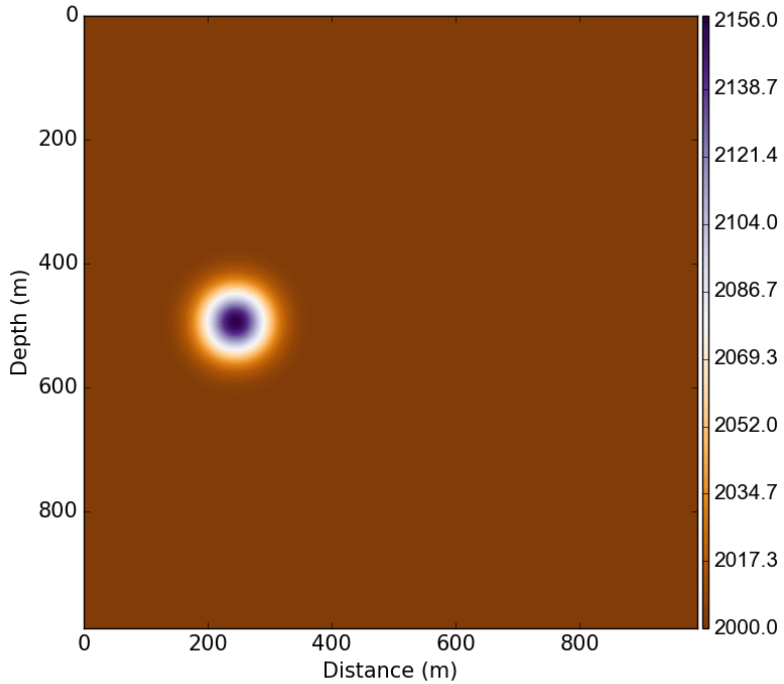


$$-\mathbf{H}_{\rho\rho}(\tilde{\mathbf{x}})$$

Multi-parameter Point Spread Function: SH Example



True Models



True P-wave Velocity: α

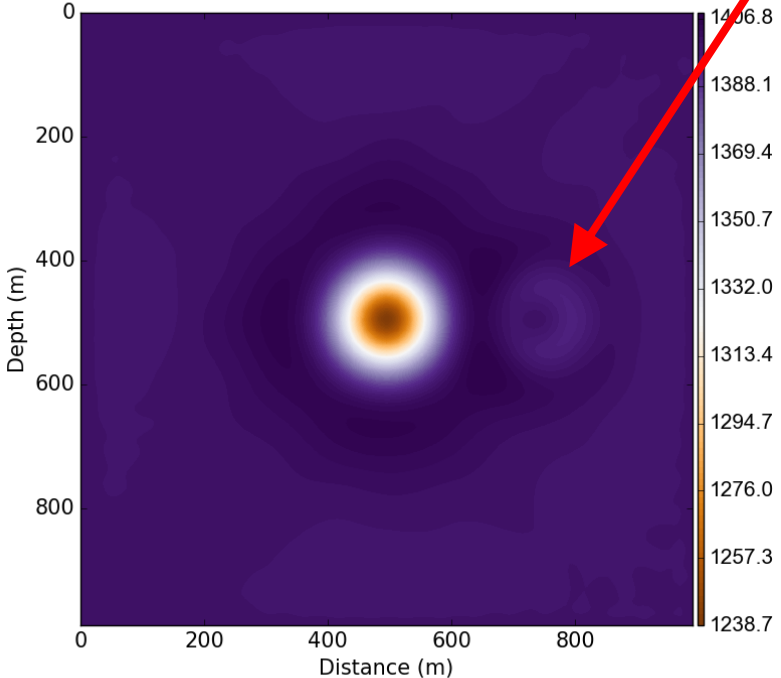
True S-wave Velocity: β

True Density: ρ

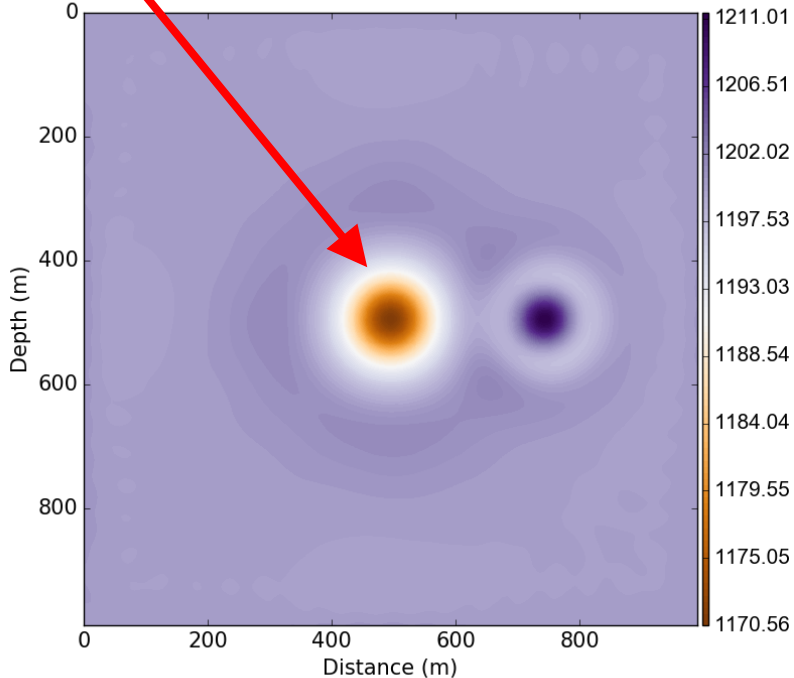
Initial Models are homogeneous.

Inverted Models

Parameter crosstalk

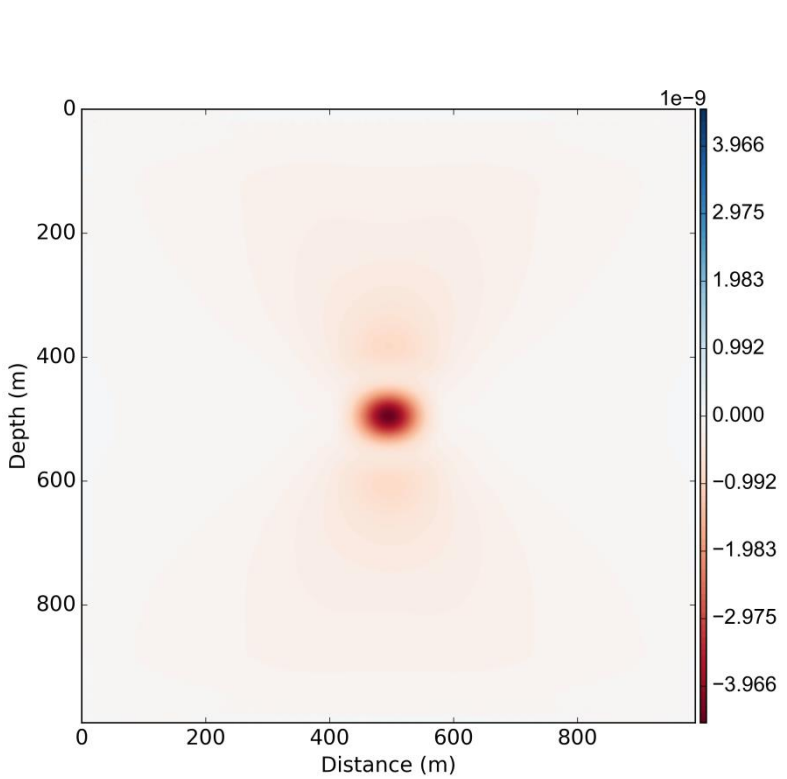


Inverted S-wave Velocity: β

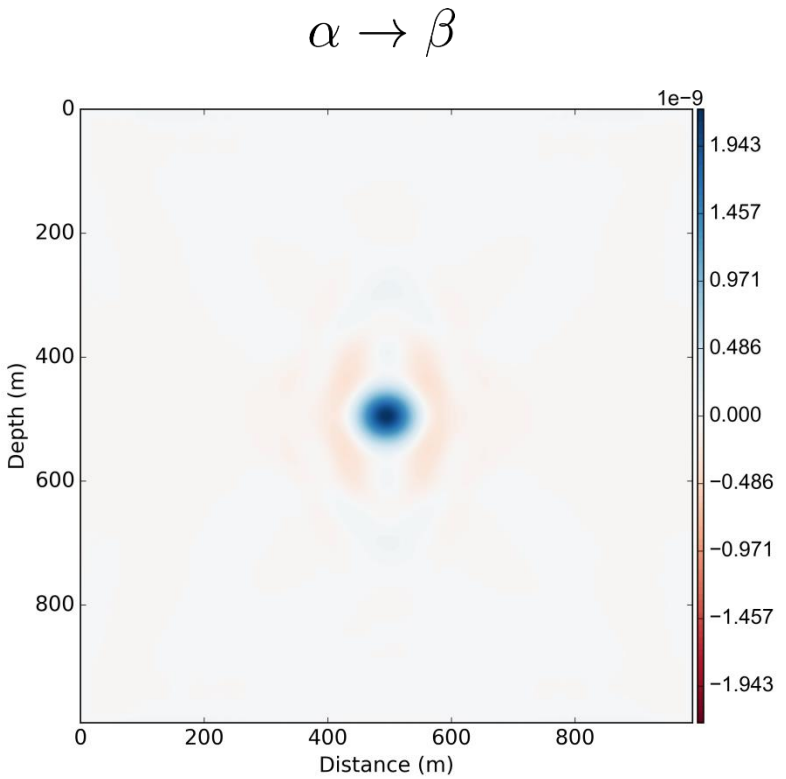


Inverted Density: ρ

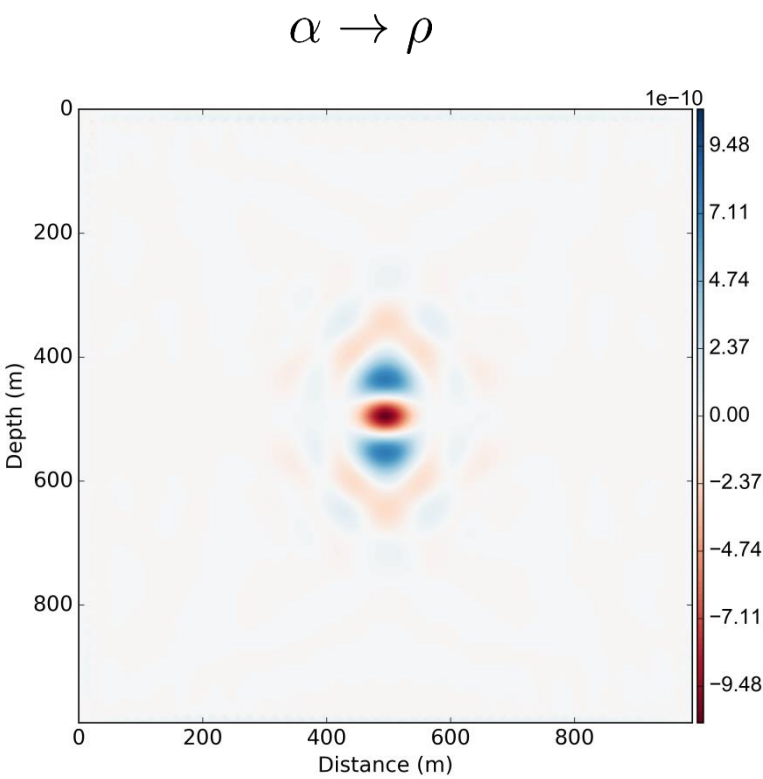
Multi-parameter Point Spread Function: P-SV Example



$-\mathbf{H}_{\alpha\alpha}(\tilde{\mathbf{x}})$



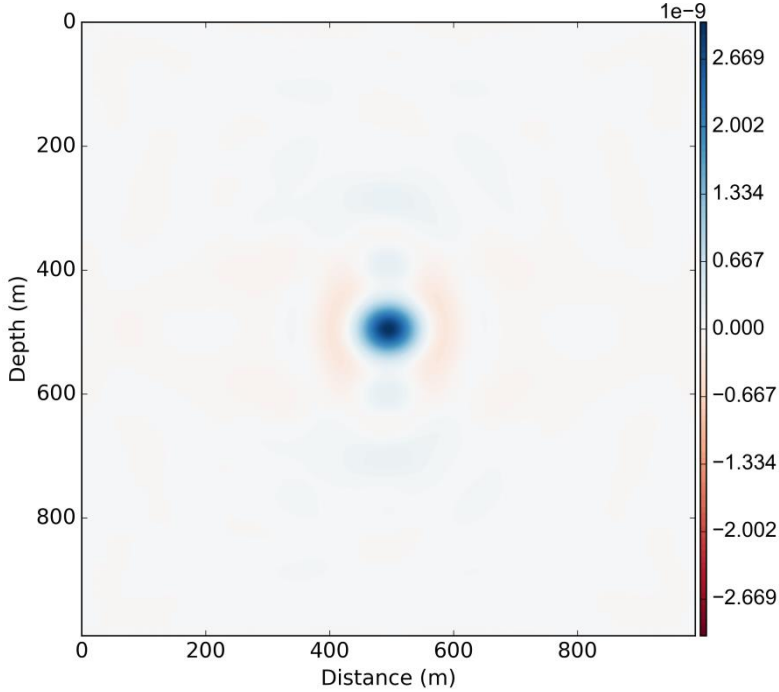
$-\mathbf{H}_{\beta\alpha}(\tilde{\mathbf{x}})$



$-\mathbf{H}_{\rho\alpha}(\tilde{\mathbf{x}})$

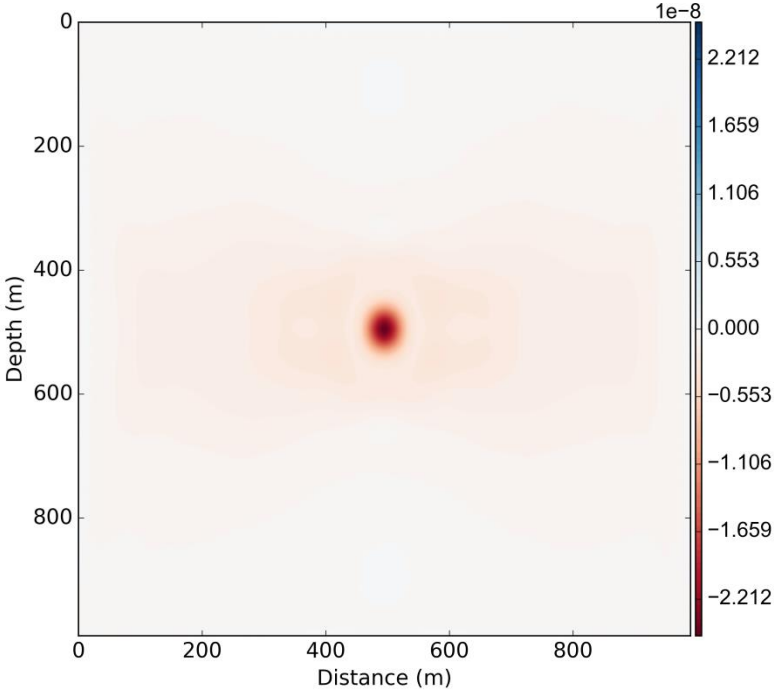
Multi-parameter Point Spread Function: P-SV Example

$$\beta \rightarrow \alpha$$

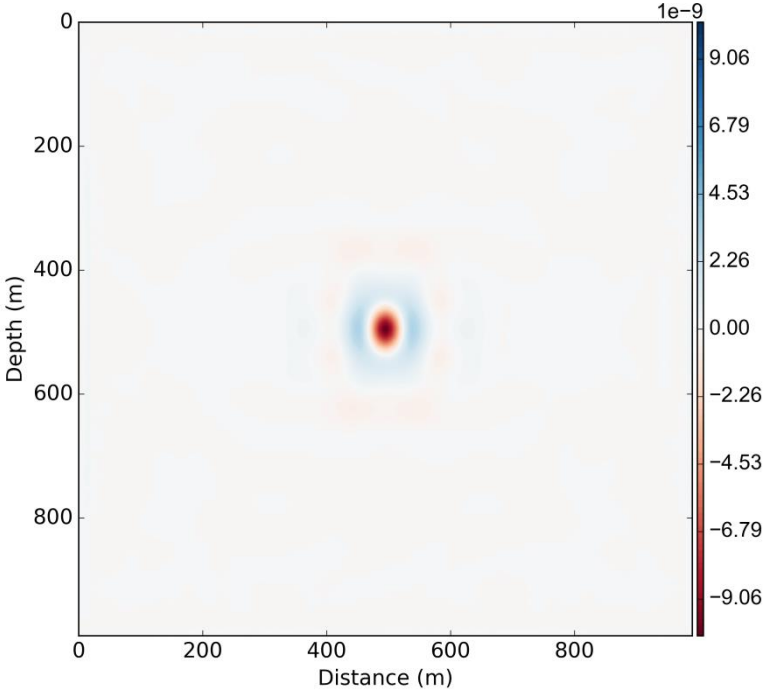


$$-\mathbf{H}_{\alpha\beta}(\tilde{\mathbf{x}})$$

$$\beta \rightarrow \rho$$



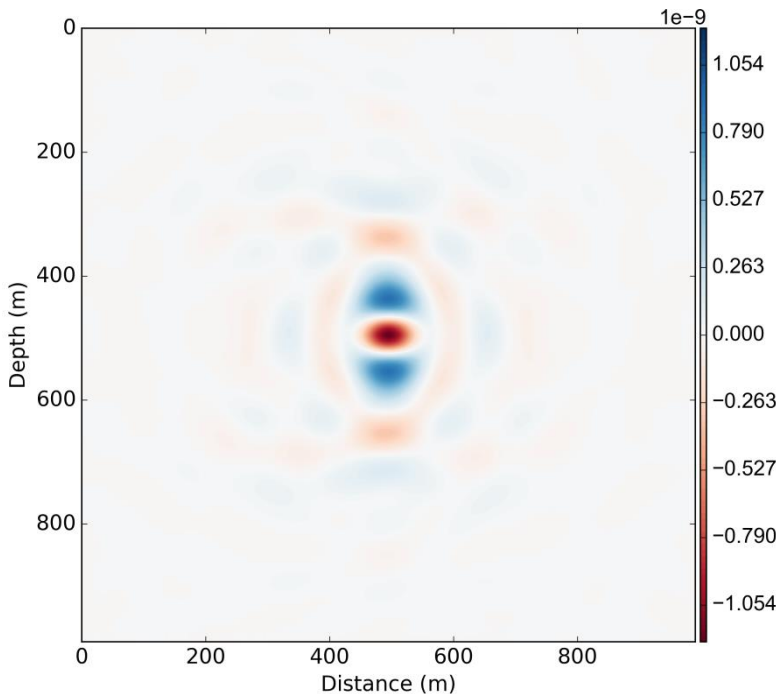
$$-\mathbf{H}_{\beta\beta}(\tilde{\mathbf{x}})$$



$$-\mathbf{H}_{\rho\beta}(\tilde{\mathbf{x}})$$

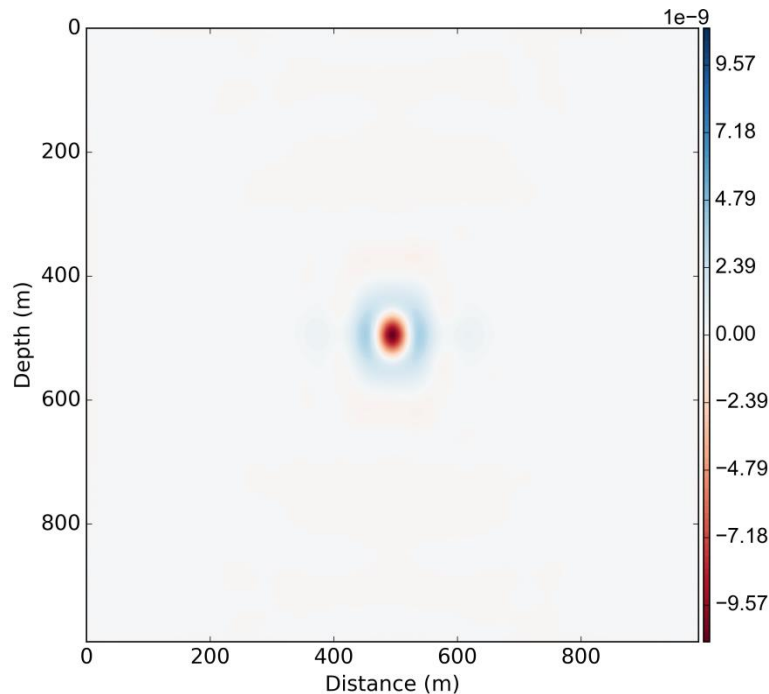
Multi-parameter Point Spread Function: P-SV Example

$\rho \rightarrow \alpha$

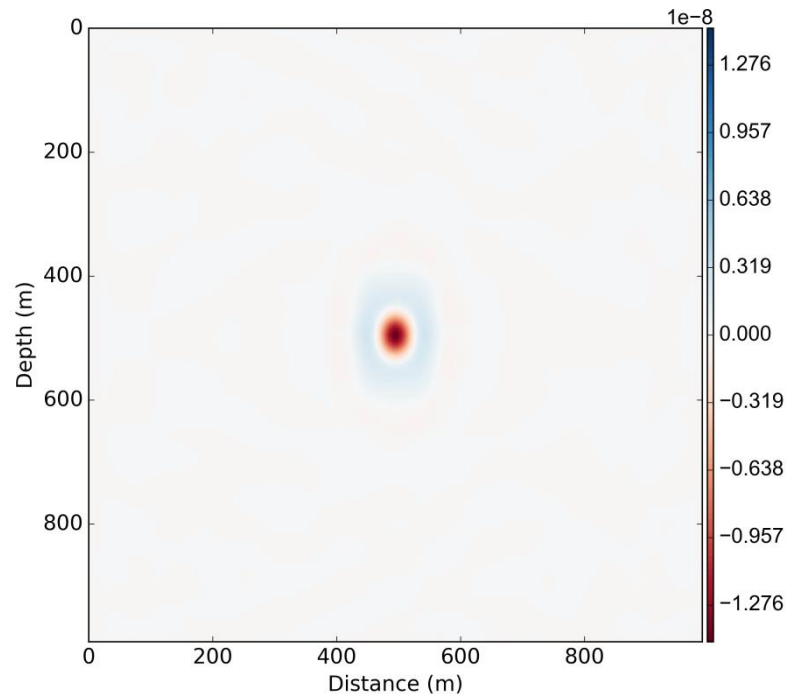


$-\mathbf{H}_{\alpha\rho}(\tilde{\mathbf{x}})$

$\rho \rightarrow \beta$

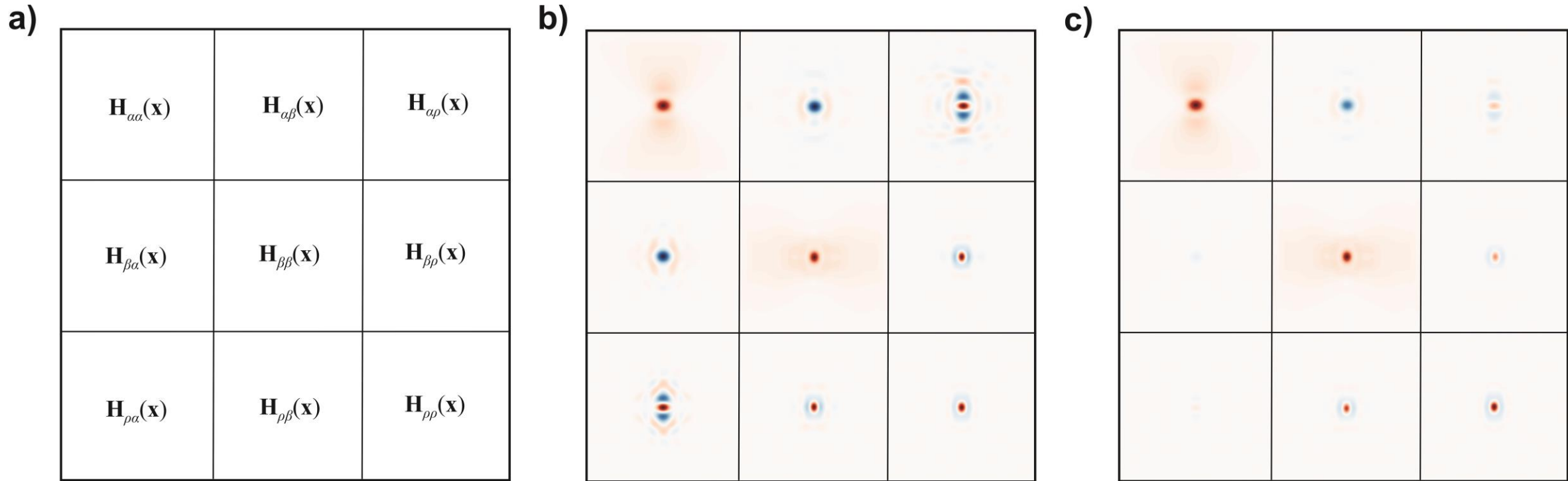


$-\mathbf{H}_{\beta\rho}(\tilde{\mathbf{x}})$



$-\mathbf{H}_{\rho\rho}(\tilde{\mathbf{x}})$

Multi-parameter Point Spread Function: P-SV Example

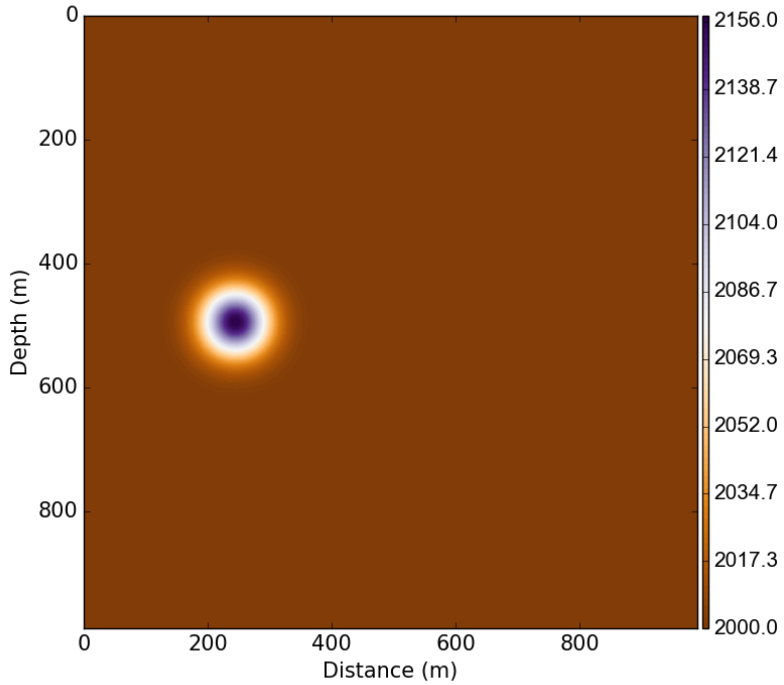


The mappings from P-wave to S-wave and density will be weak;

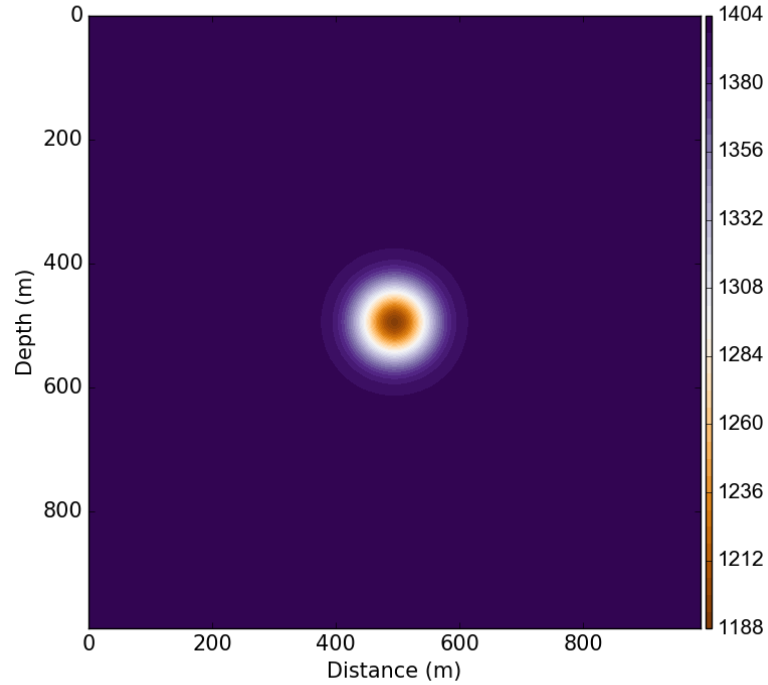
The mapping of positive S-wave velocity perturbation to P-wave velocity is negative and vice versa;

The mappings from S-wave velocity to P-wave velocity and density are stronger;

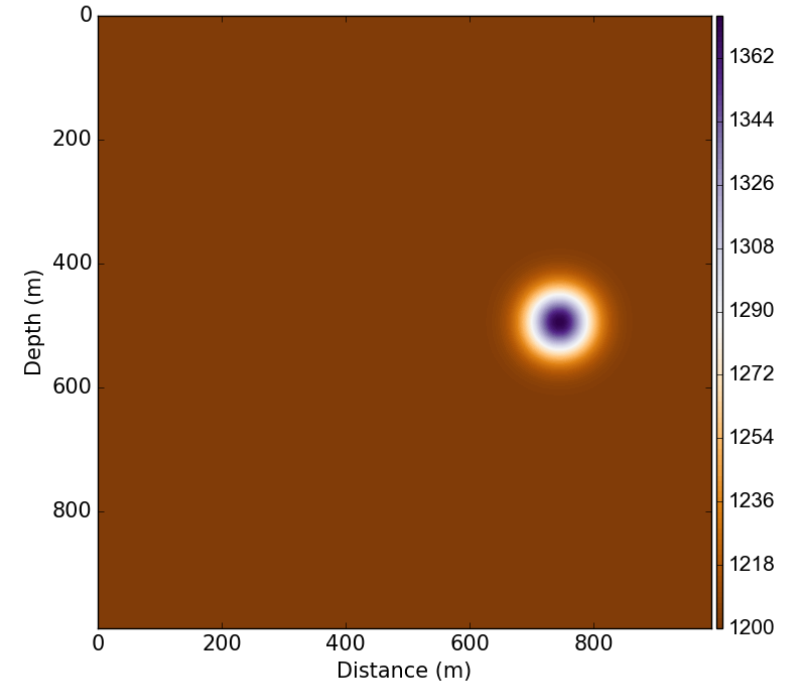
True Models



True P-wave Velocity: α



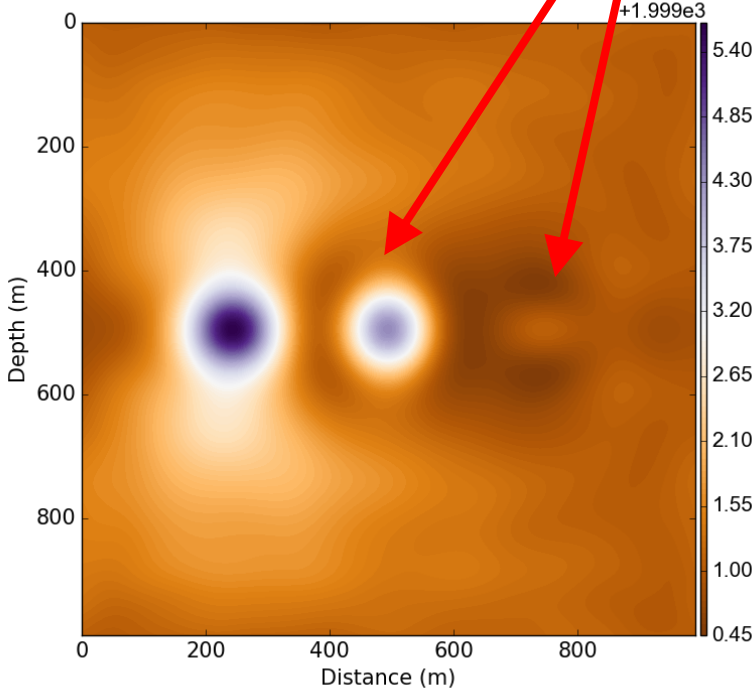
True S-wave Velocity: β



True Density: ρ

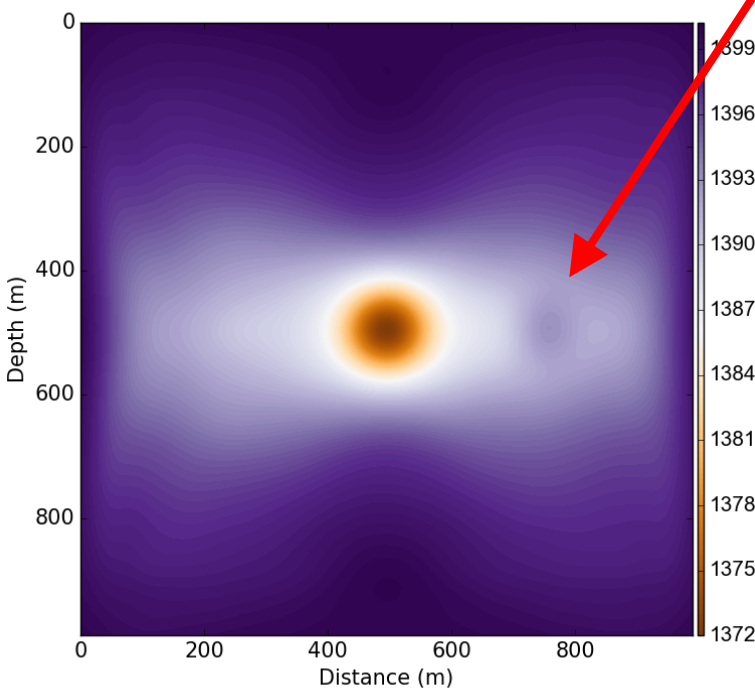
Inverted Models: P-SV Example 1st Iteration

Parameter crosstalk

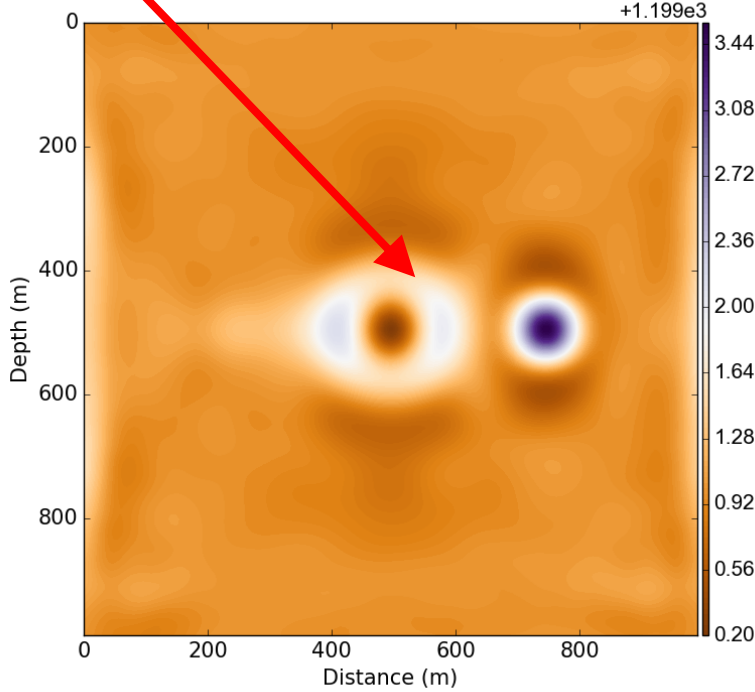


Inverted P-wave Velocity: α

Parameter crosstalk



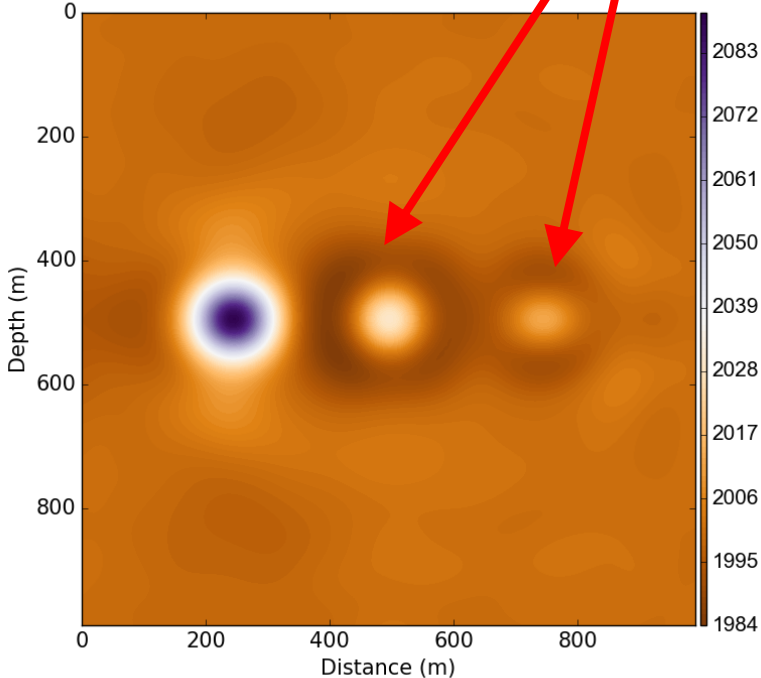
Inverted S-wave Velocity: β



Inverted Density: ρ

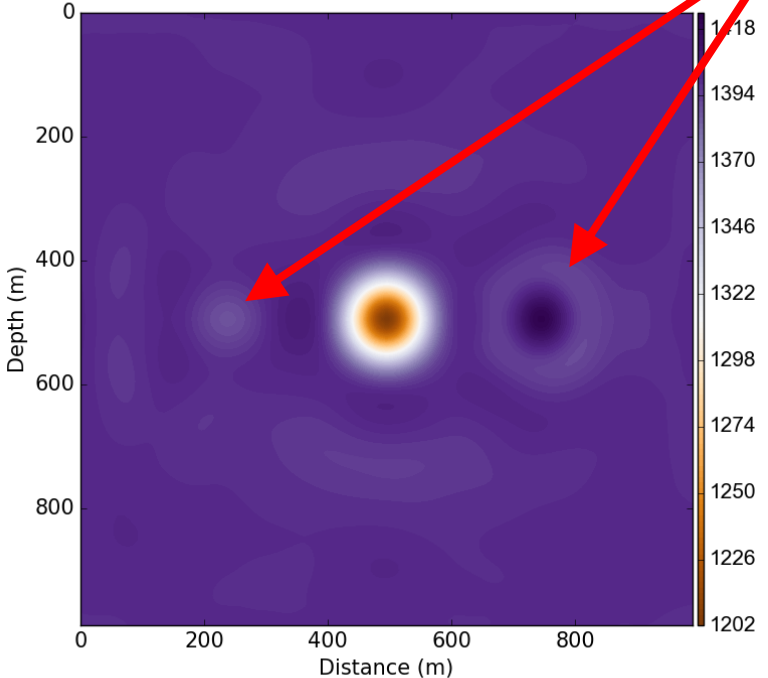
Inverted Models: P-SV Example 6th Iteration

Parameter crosstalk

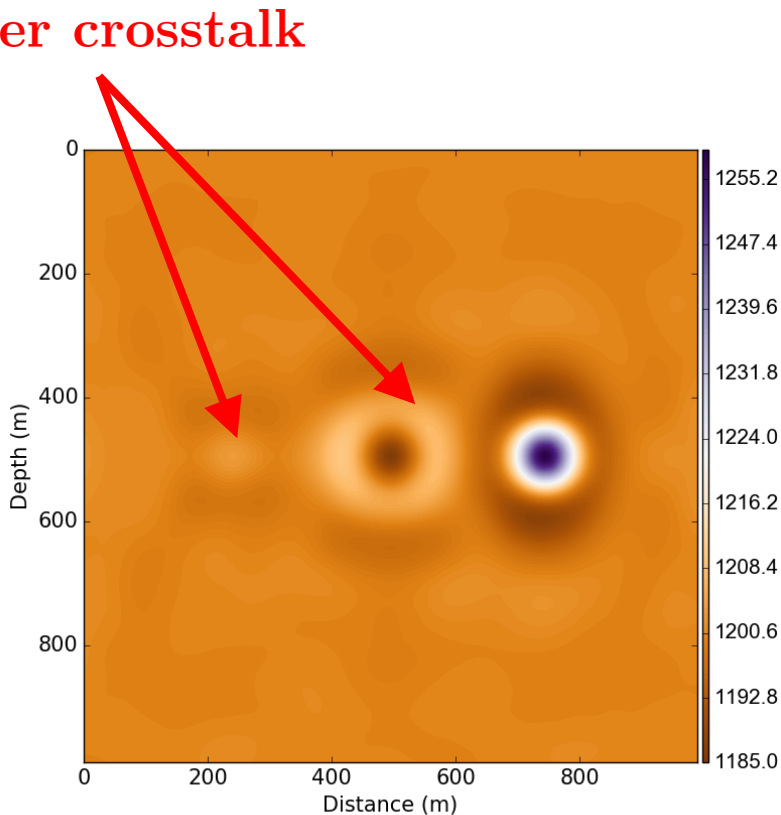


Inverted P-wave Velocity: α

Parameter crosstalk

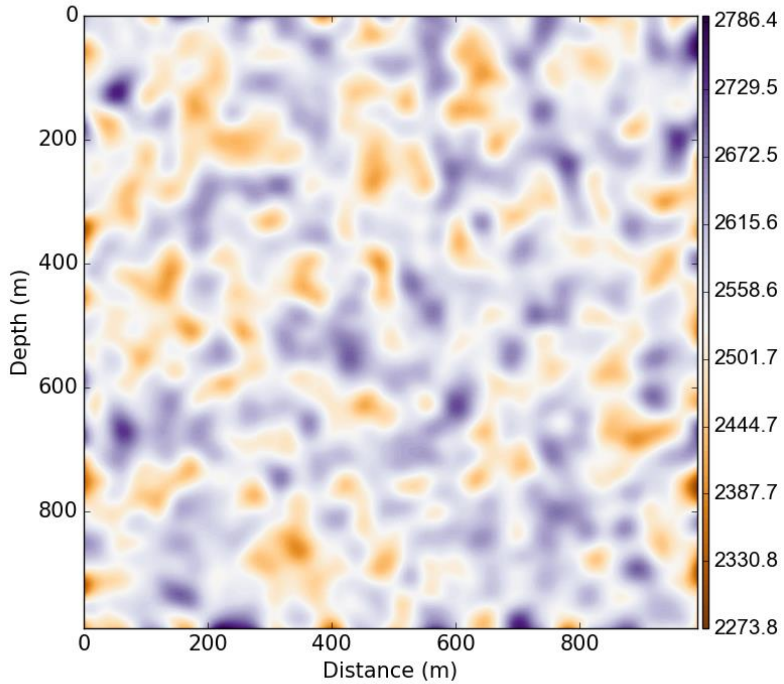


Inverted S-wave Velocity: β

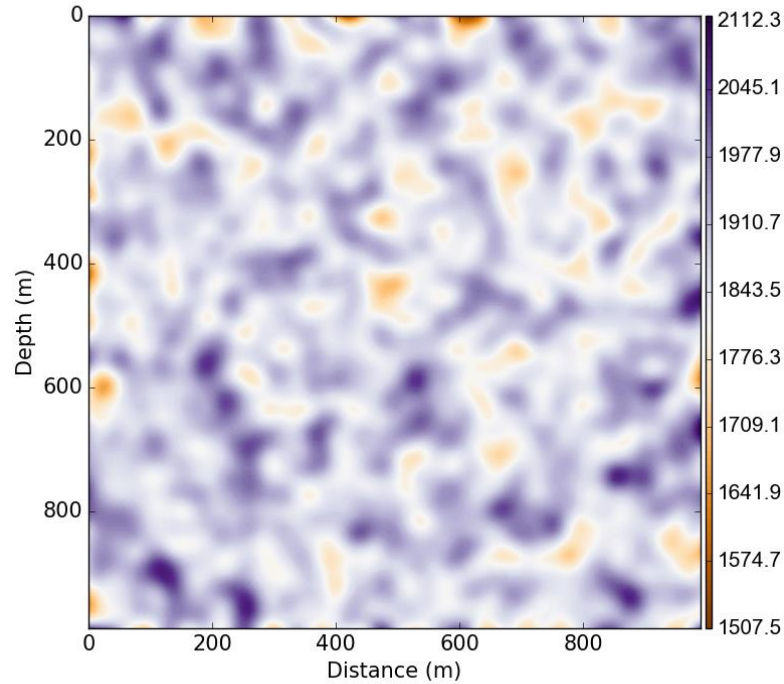


Inverted Density: ρ

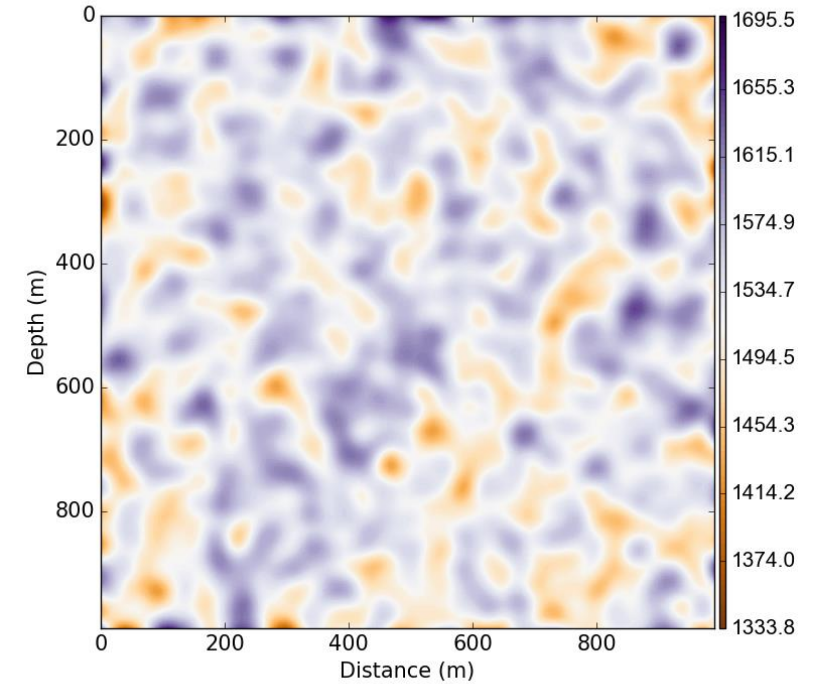
True Models: Random Media Example



True P-wave Velocity: α

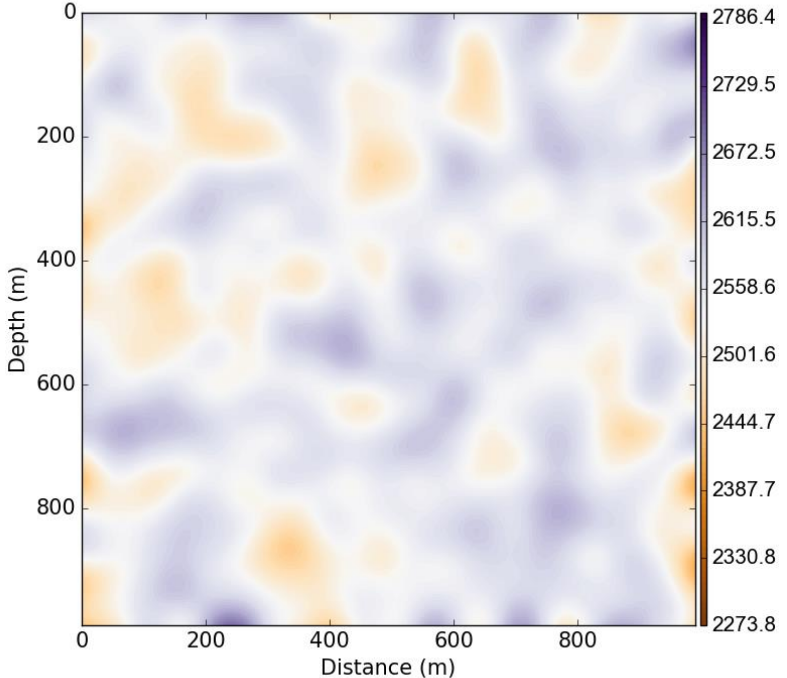


True S-wave Velocity: β

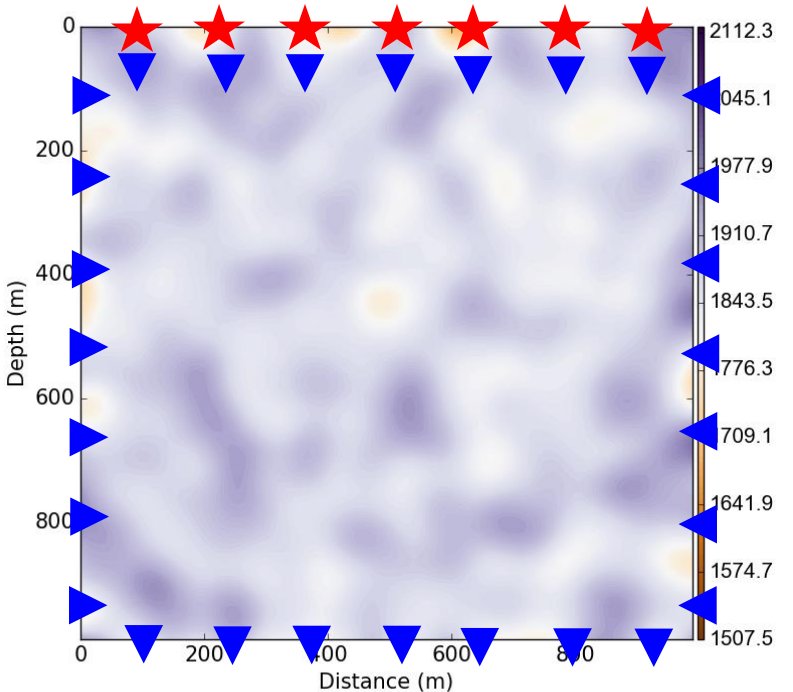


True Density: ρ

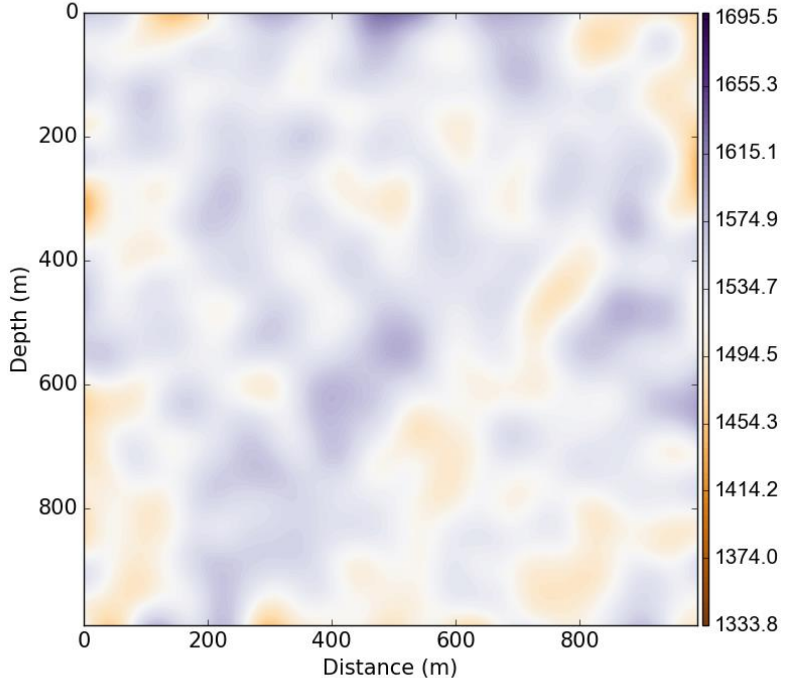
Initial Models



Initial P-wave Velocity: α

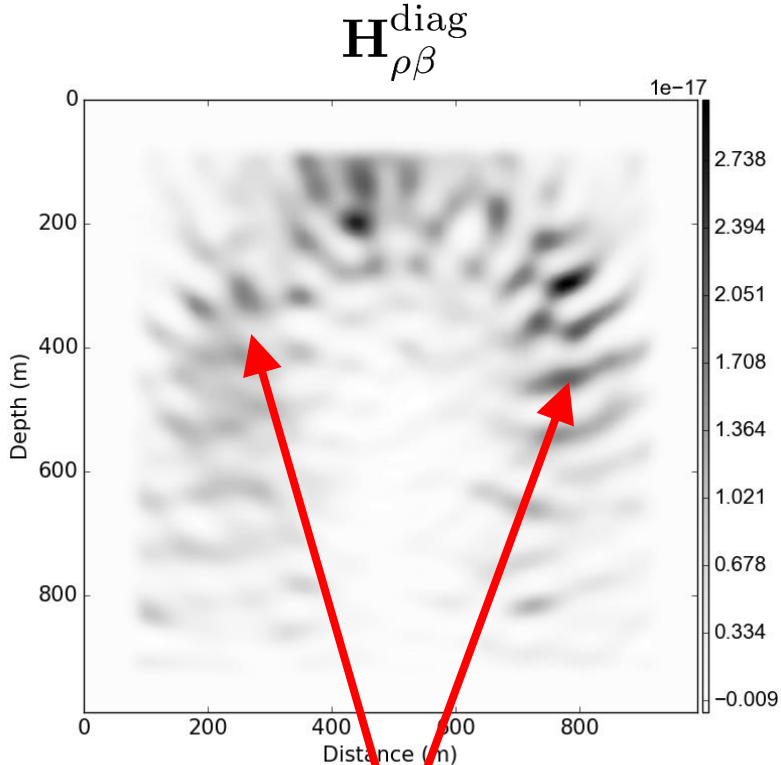
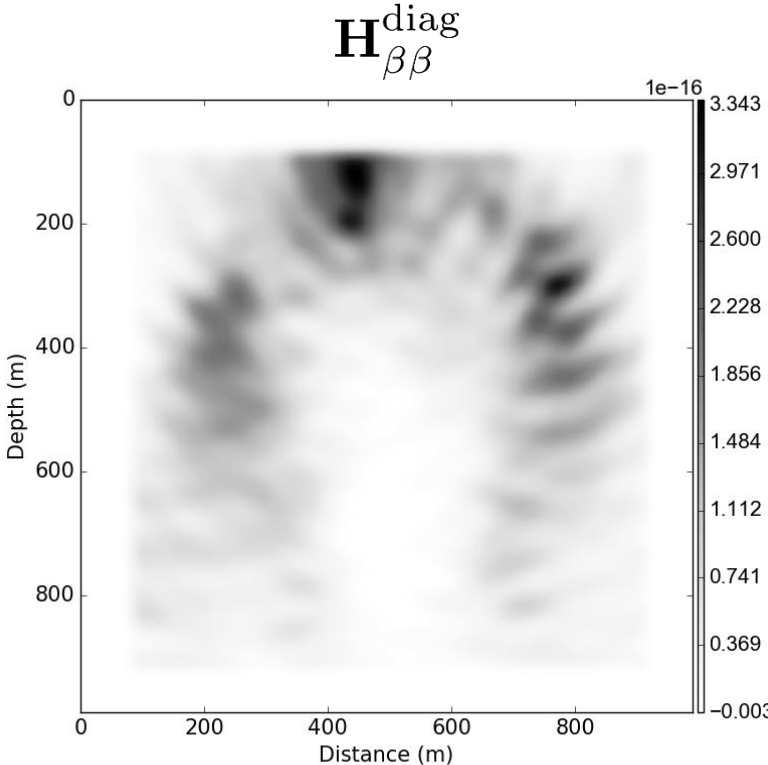


Initial S-wave Velocity: β



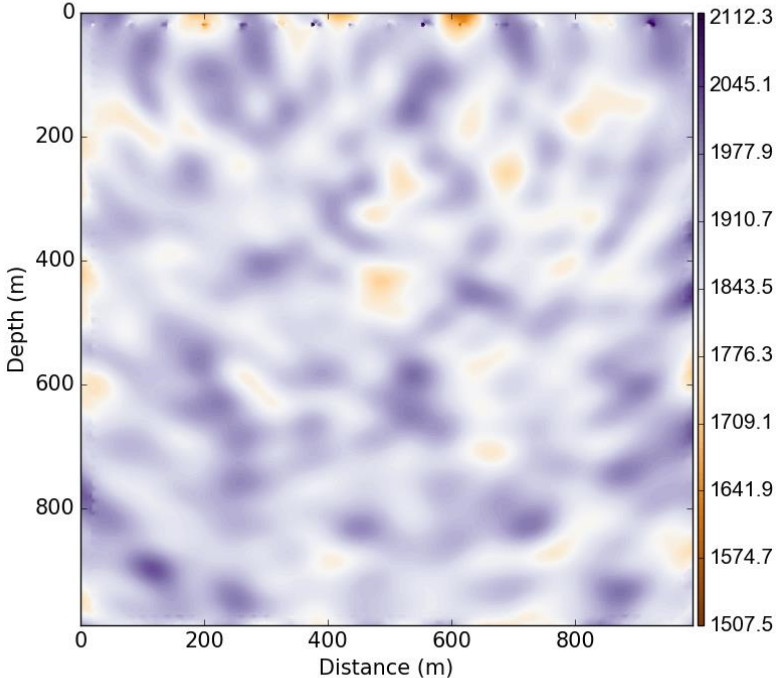
Initial Density: ρ

Stochastic Probing: SH Example

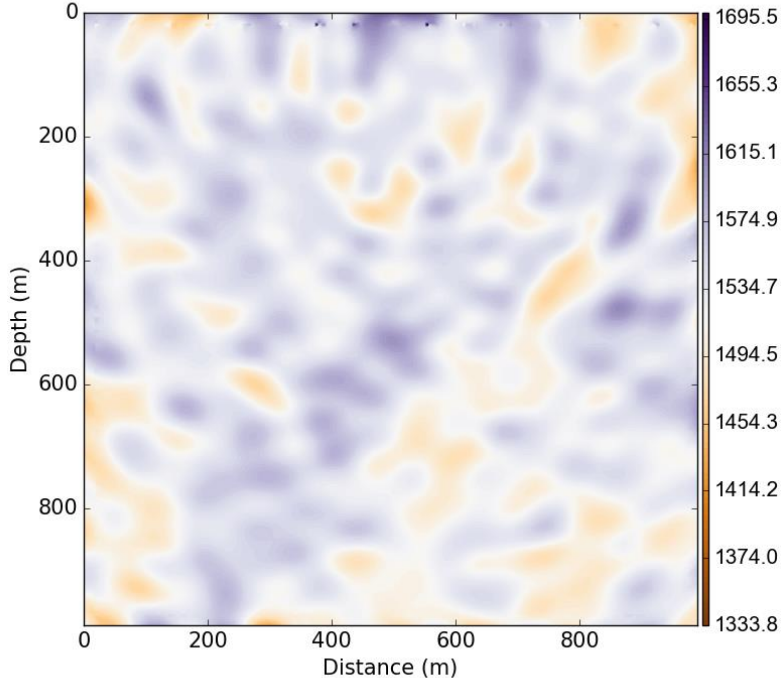


Strong Parameter crosstalk Area

Inverted Models: SH Example

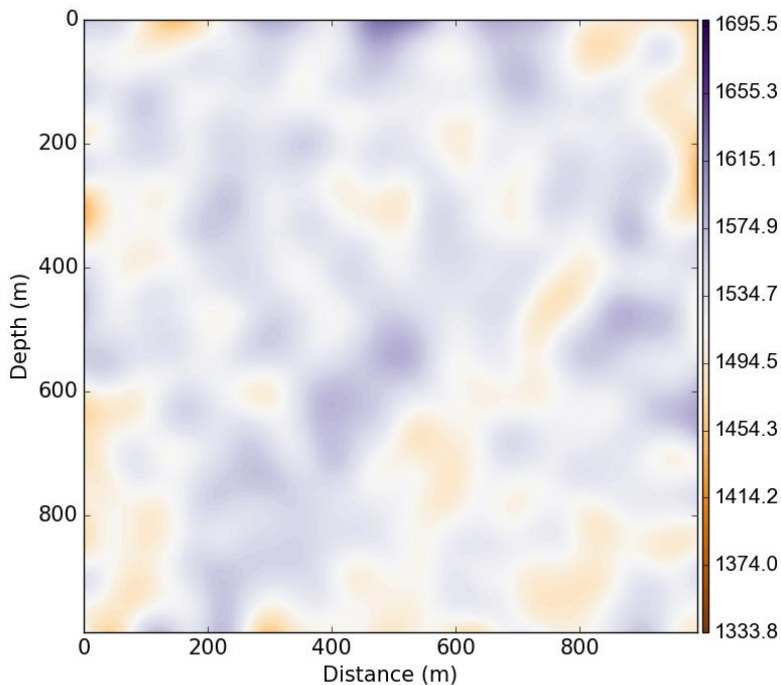
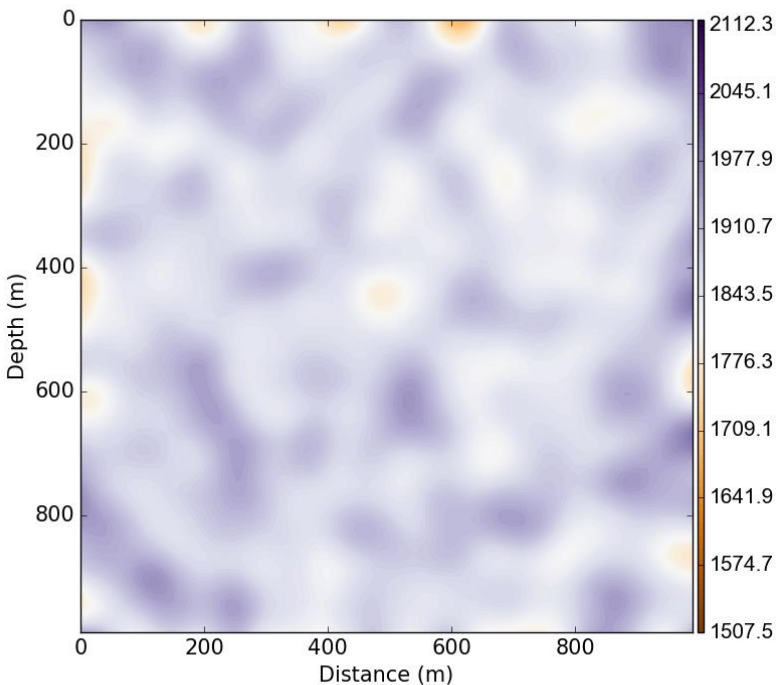
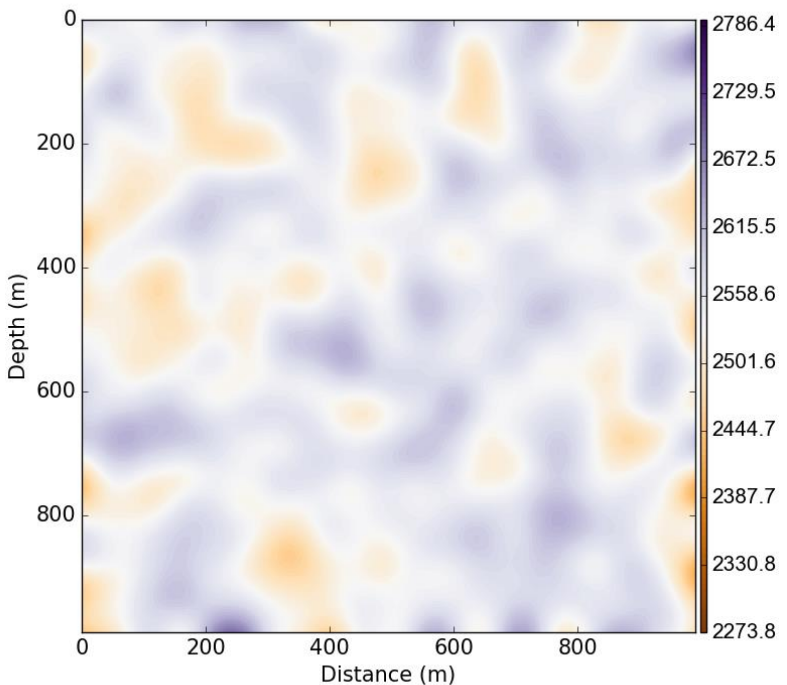


Inverted S-wave Velocity: β



Inverted Density: ρ

Initial Models

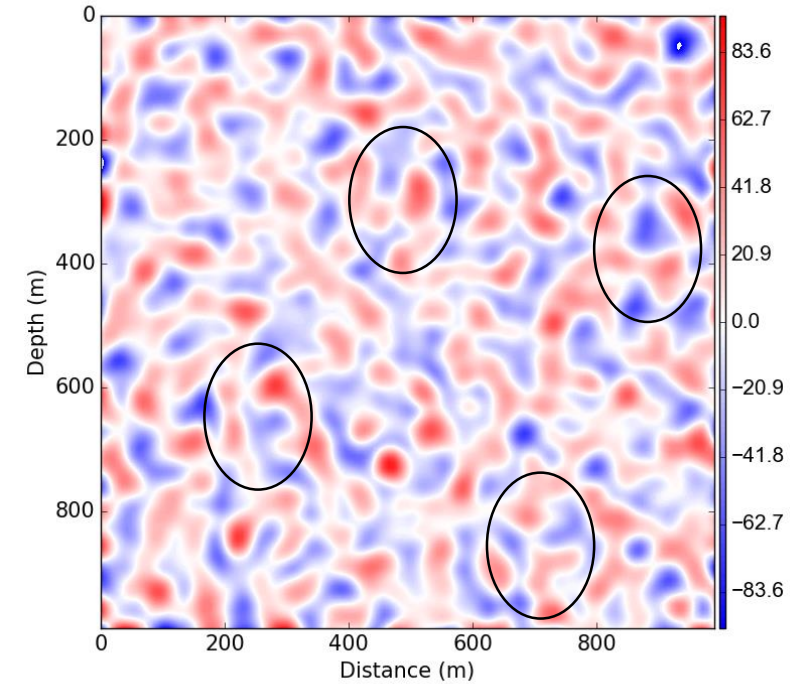
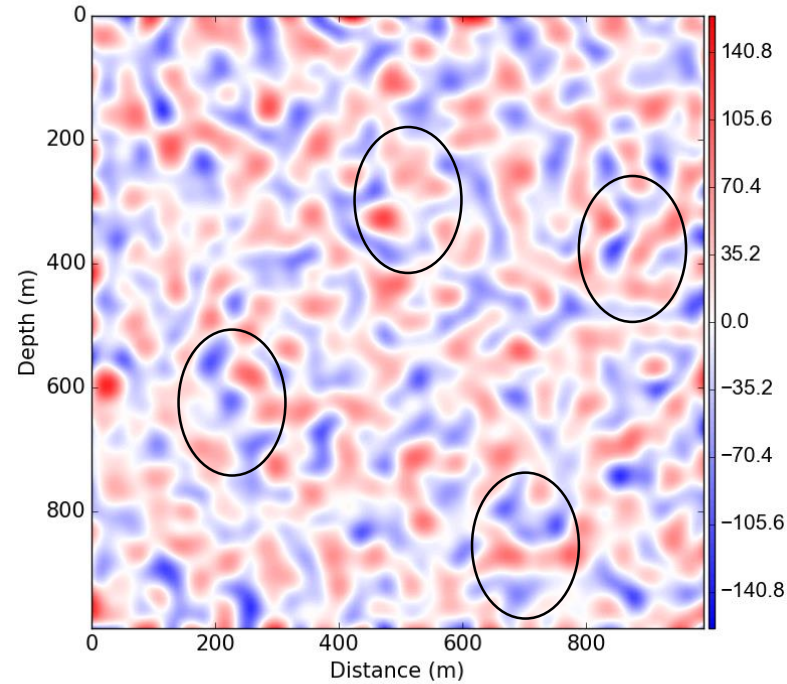


Initial P-wave Velocity: α

Initial S-wave Velocity: β

Initial Density: ρ

True Model Perturbations

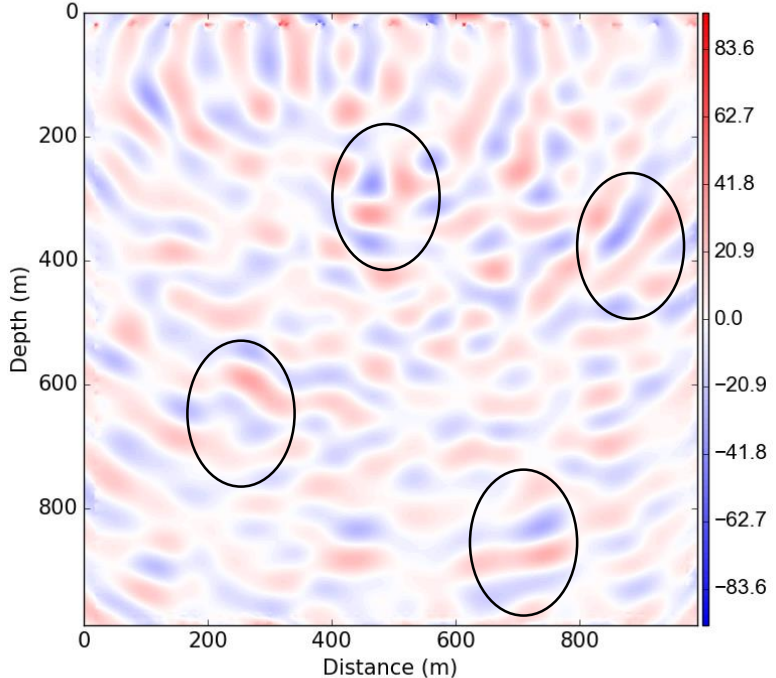
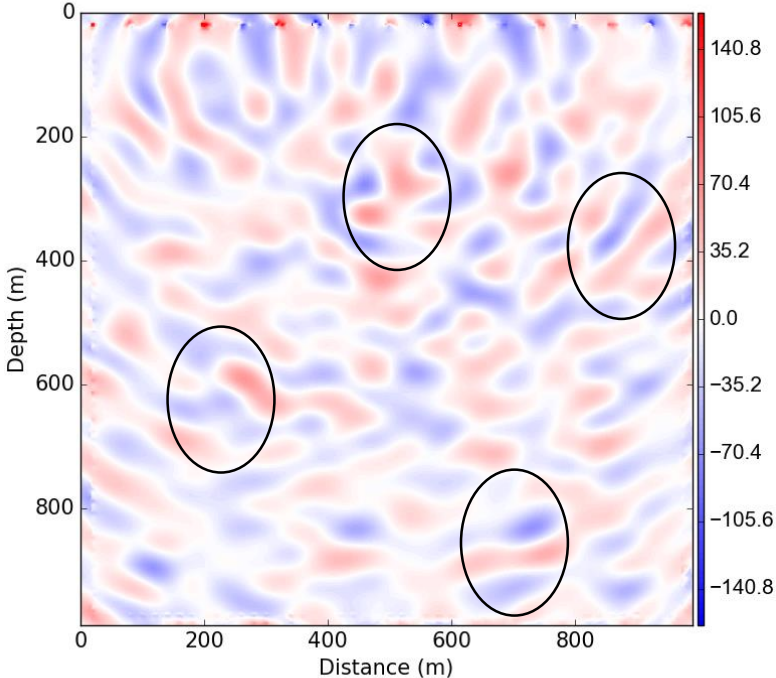


True S-wave Velocity Perturbation True Density Perturbation

Estimated Model Perturbations

Inverted density perturbation is very similar to S-wave velocity perturbation.

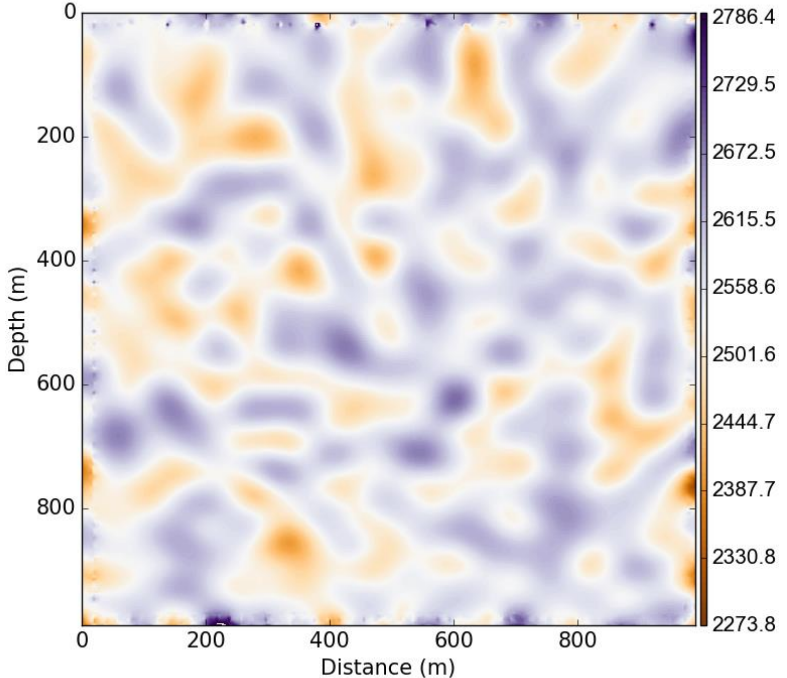
Strong interparameter mapping from S-wave velocity makes density difficult to be inverted.



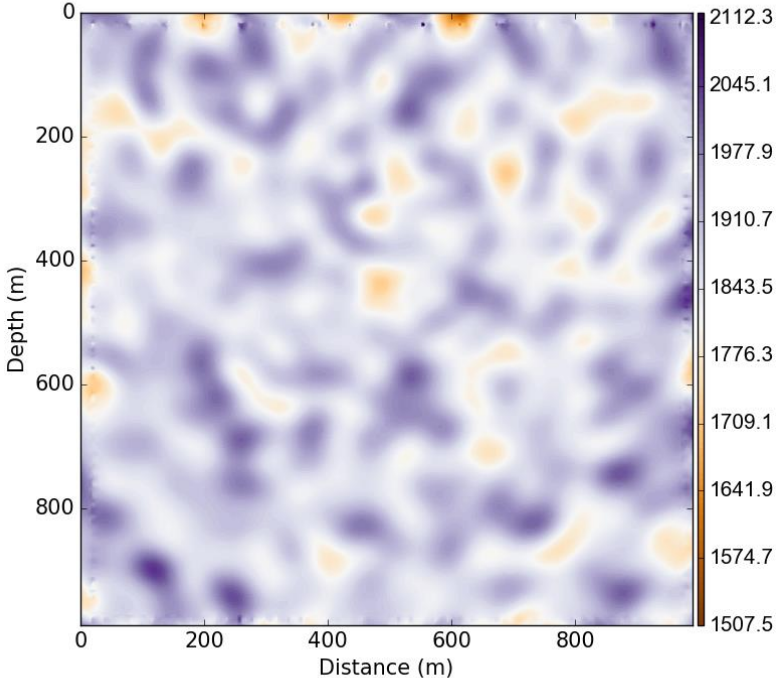
Inverted S-wave Velocity Perturbation

Inverted Density Perturbation

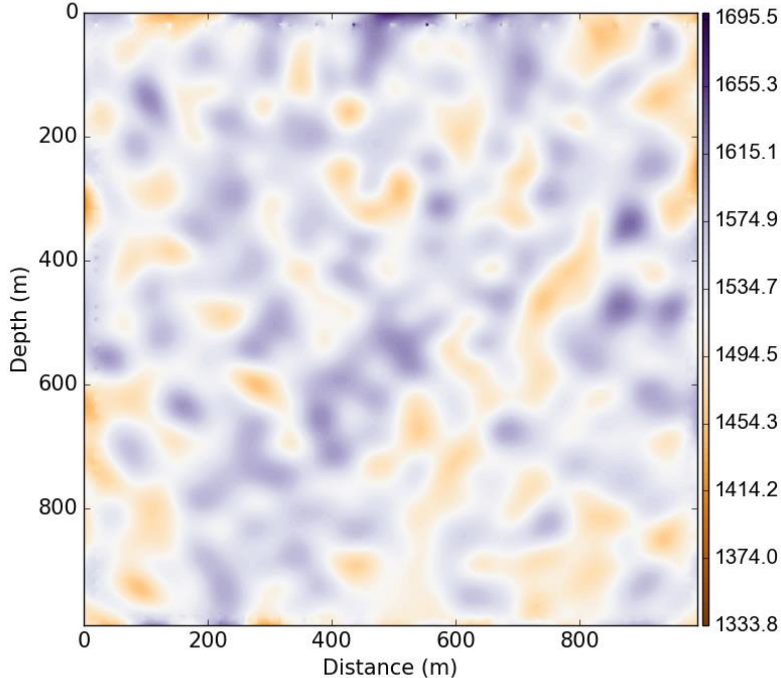
Inverted Models: P-SV Example



Inverted P-wave Velocity: α

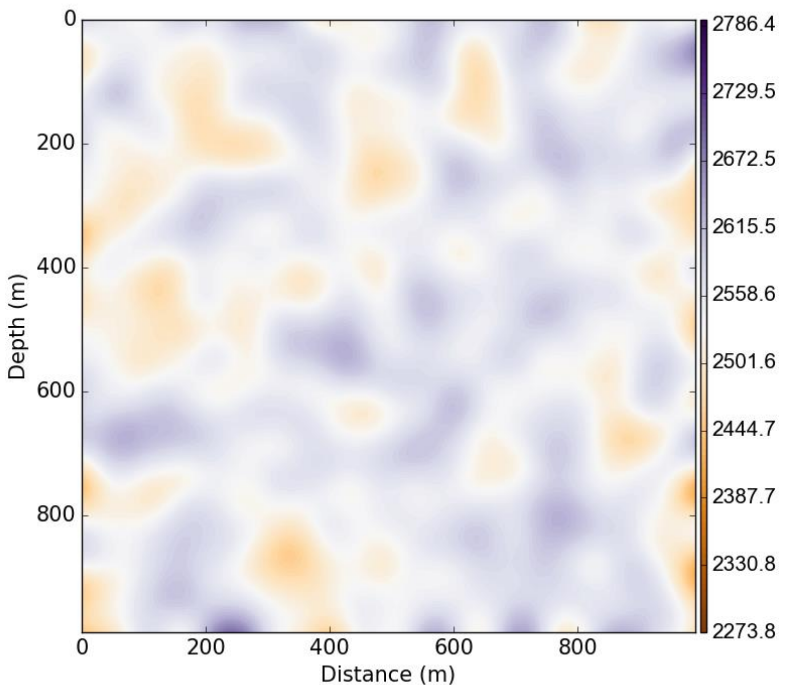


Inverted S-wave Velocity: β

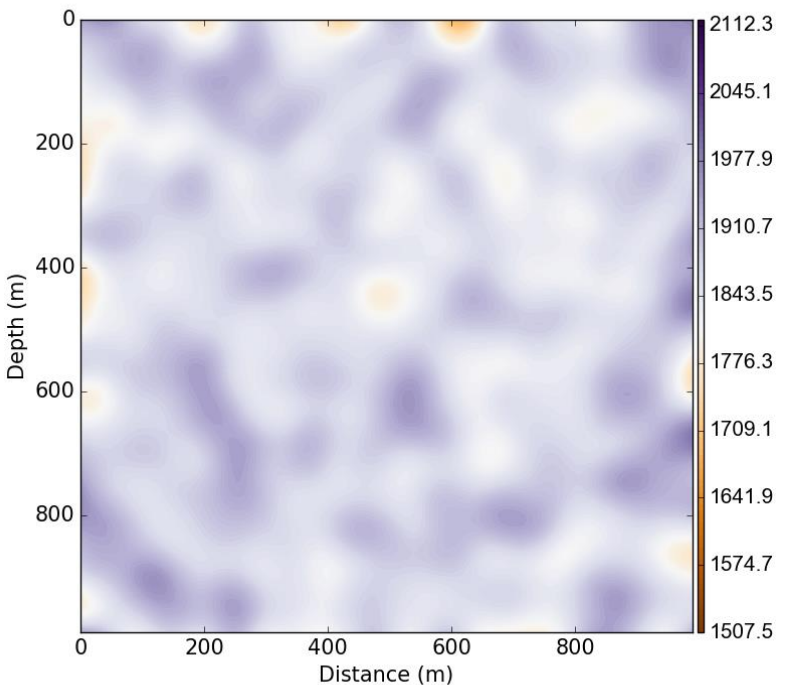


Inverted Density: ρ

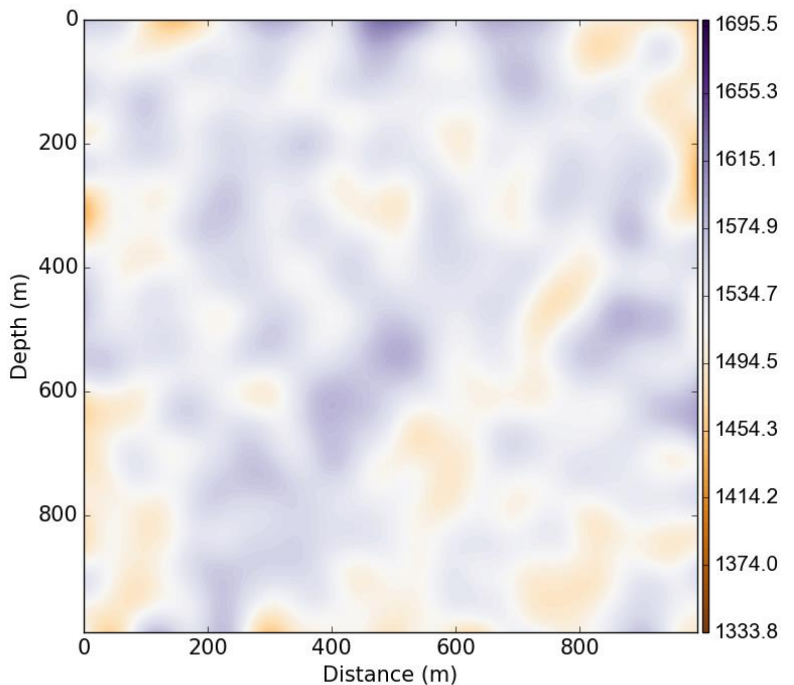
Initial Models



Initial P-wave Velocity: α

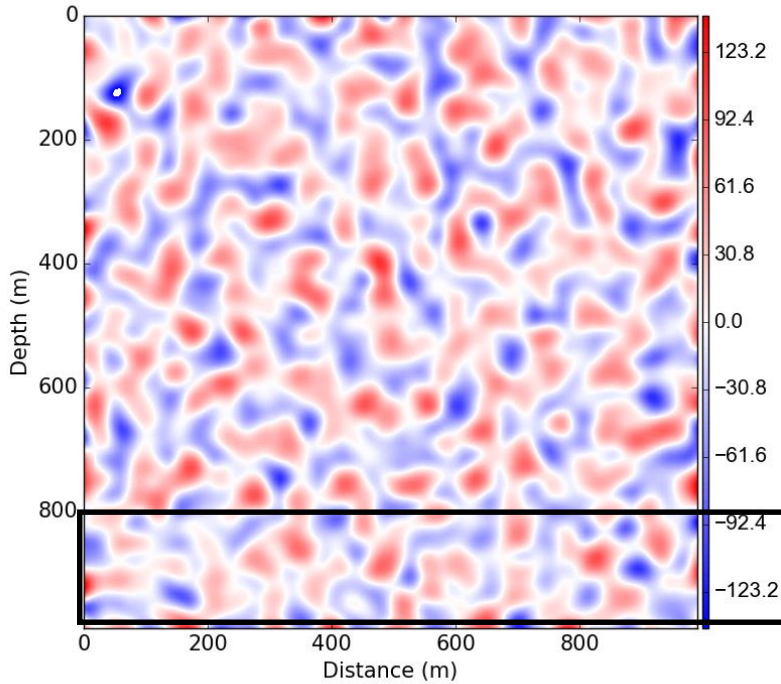


Initial S-wave Velocity: β

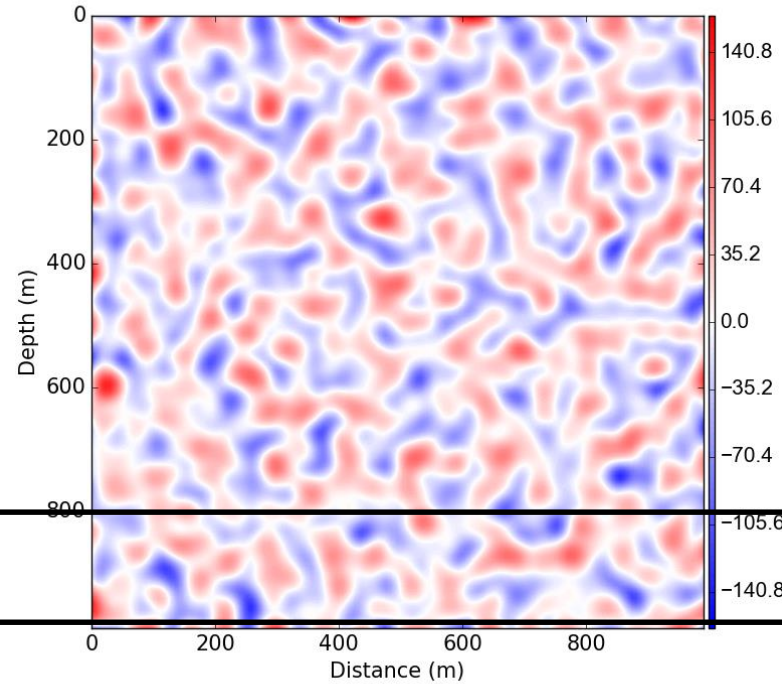


Initial Density: ρ

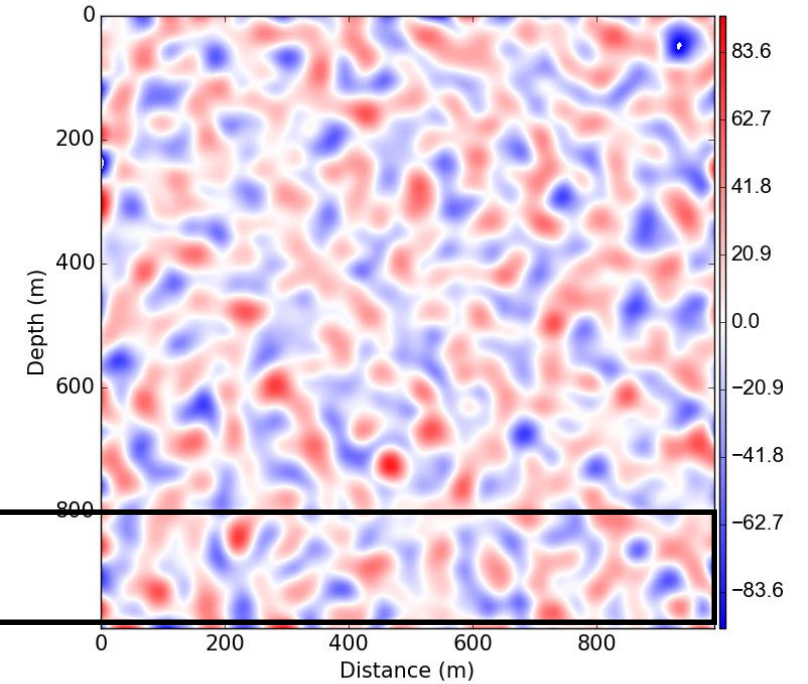
True Model Perturbations



True α Perturbation

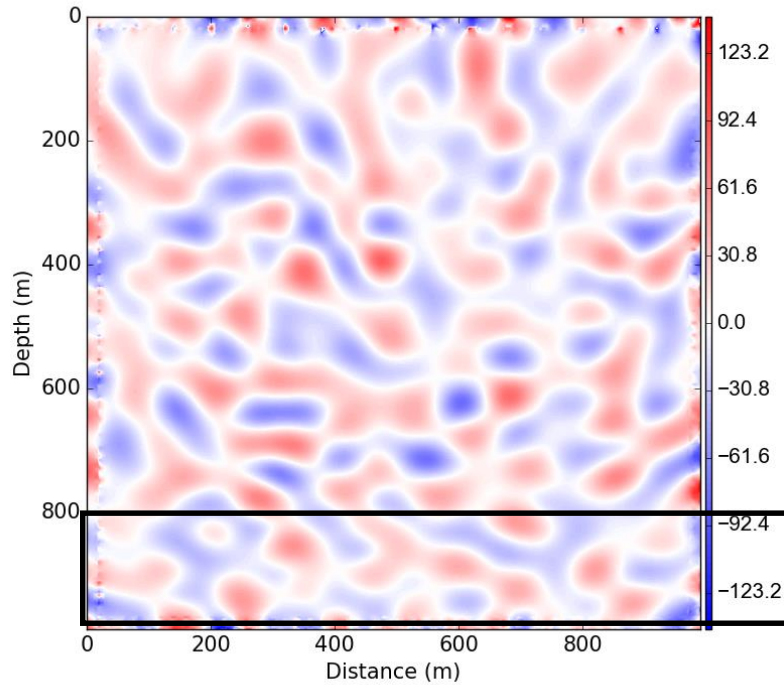


True β Perturbation

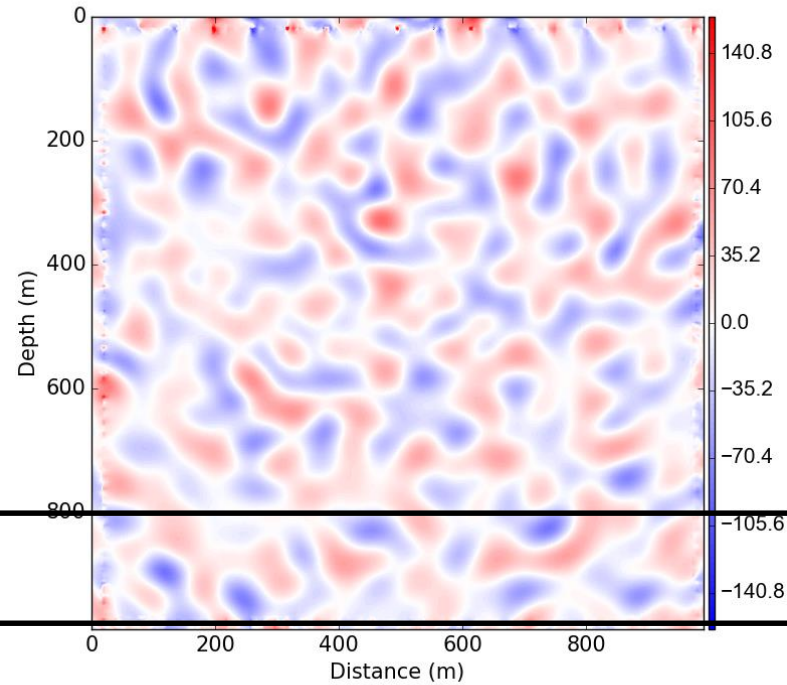


True ρ Perturbation

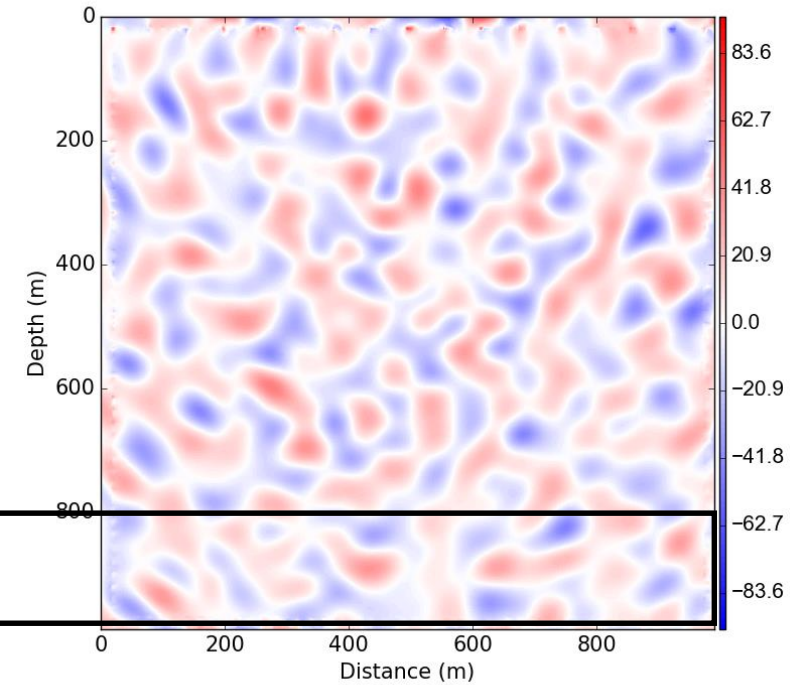
Inverted Model Perturbations



Inverted α Perturbation



Inverted β Perturbation



Inverted ρ Perturbation

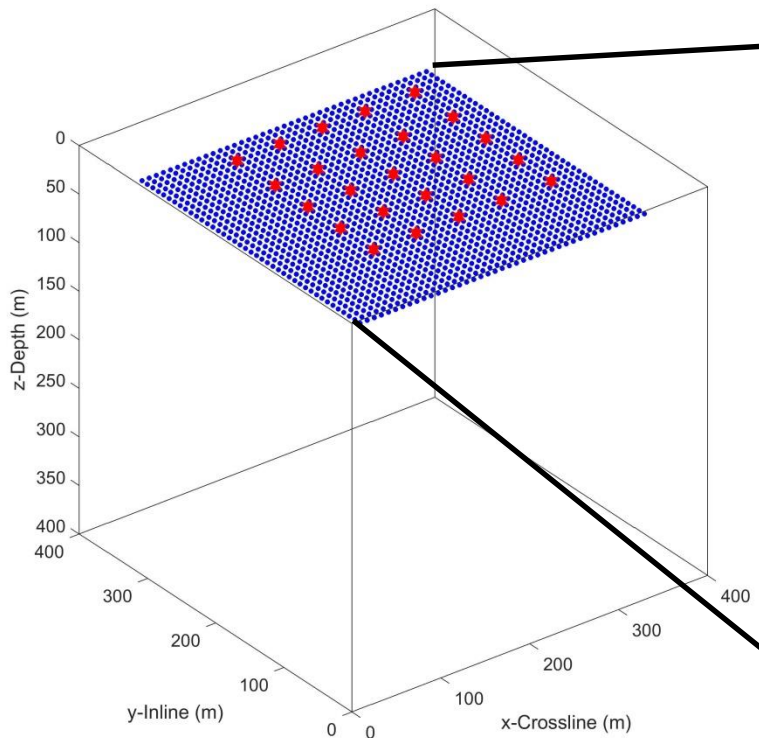
S-wave velocity is best inverted;

P-wave velocity and density are contaminated by parameter crosstalk artifacts;

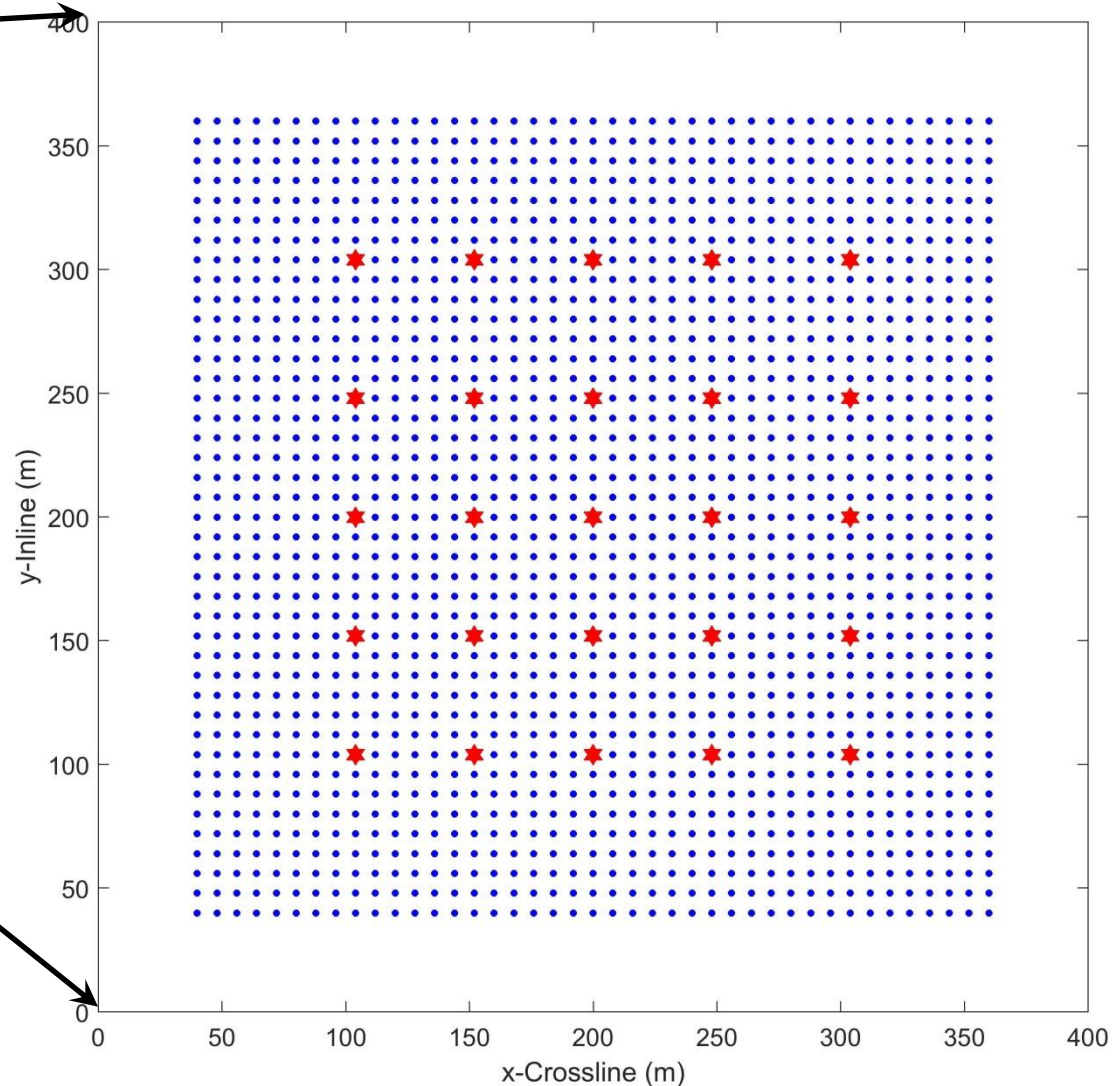
4.2 Anisotropic and Elastic Example

Multi-parameter Point Spread Function in Anisotropic Media

★ source ● receiver

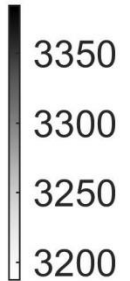
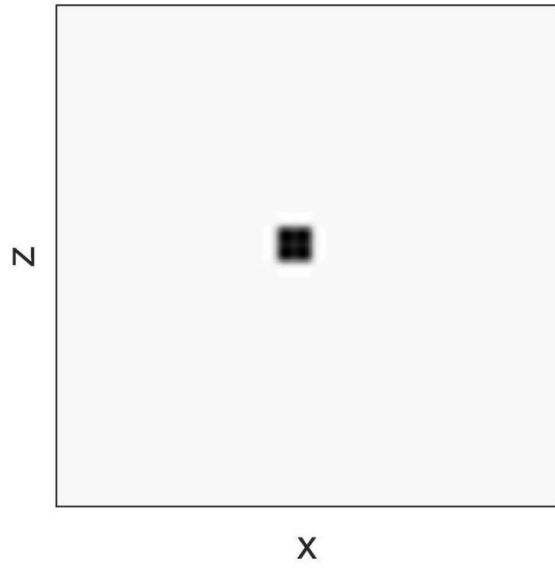
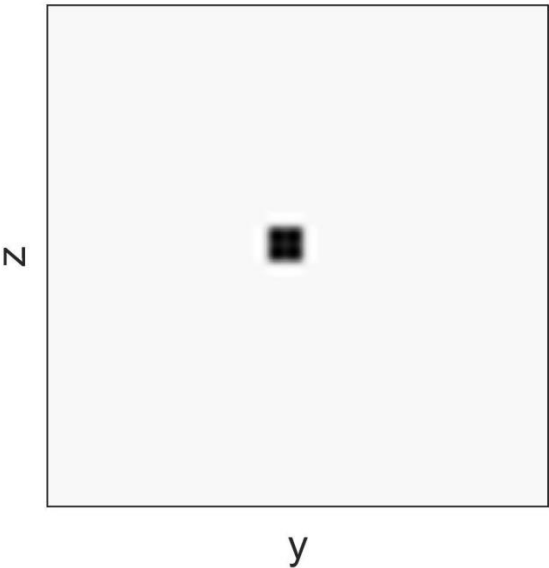
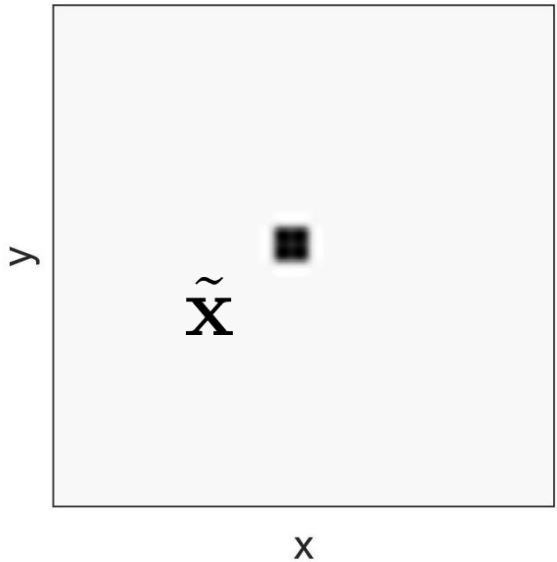
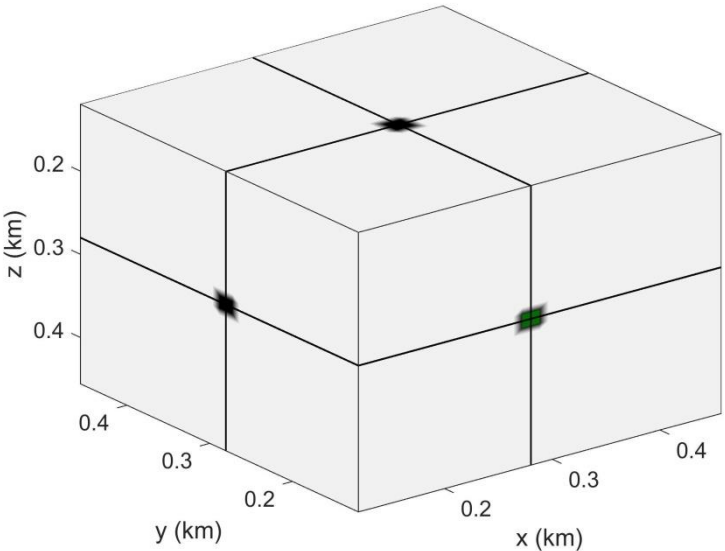


★ source ● receiver

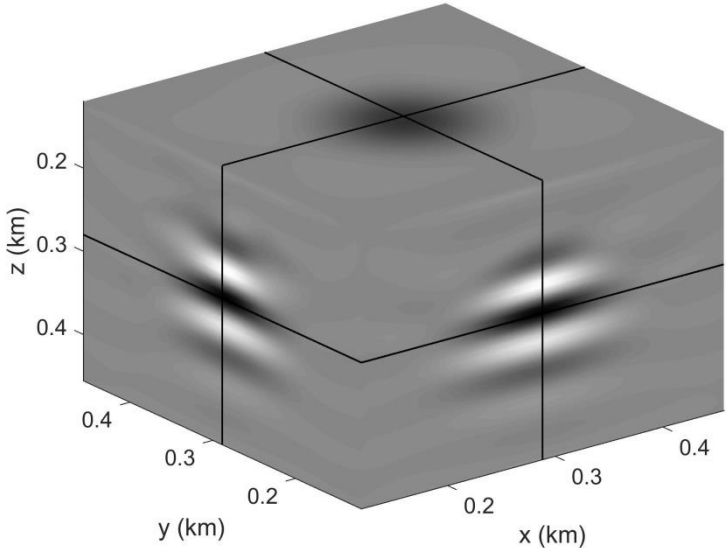


3D Homogeneous TI Media

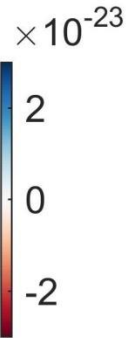
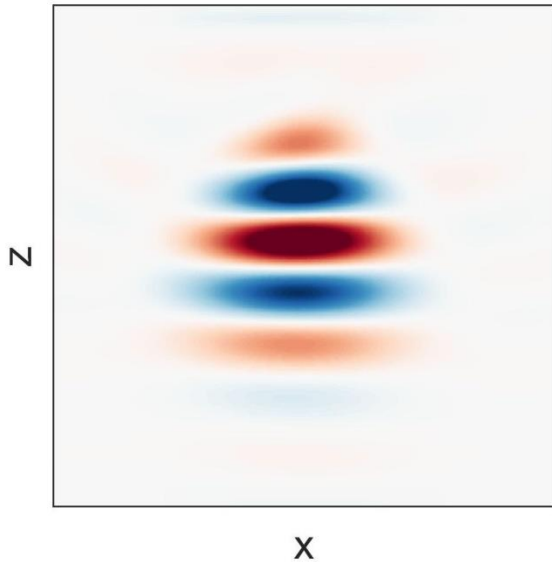
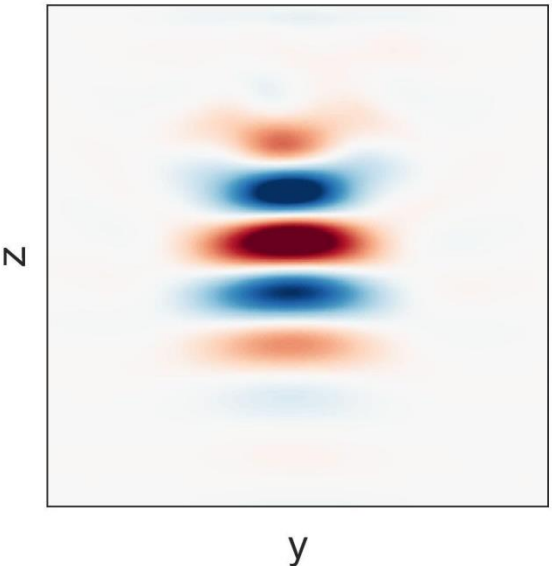
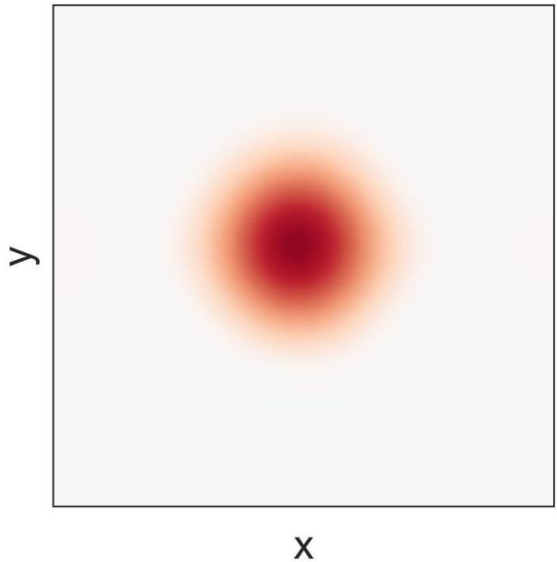
Numerical Examples



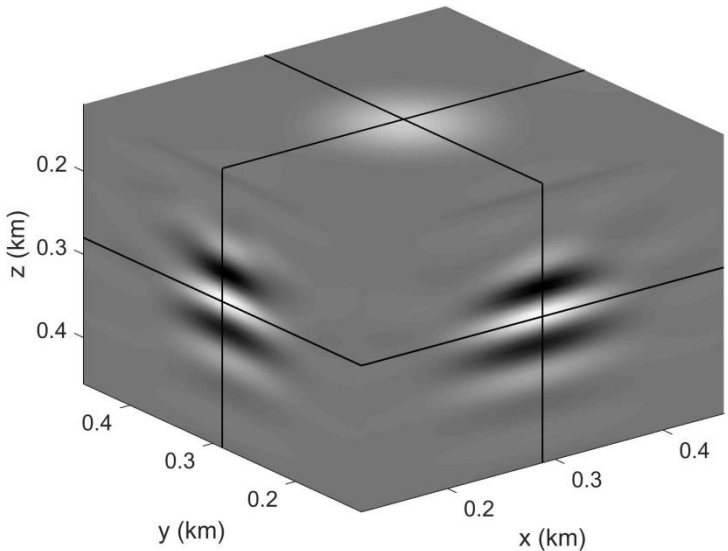
Multi-parameter Point Spread Function in Anisotropic Media



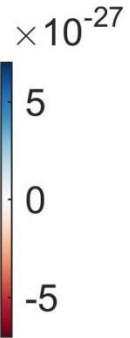
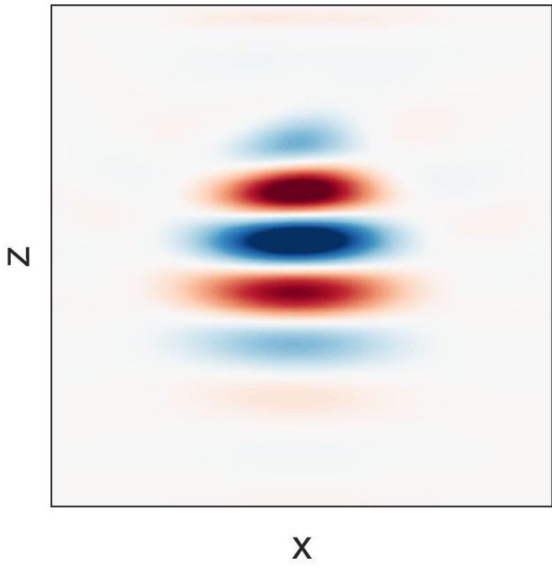
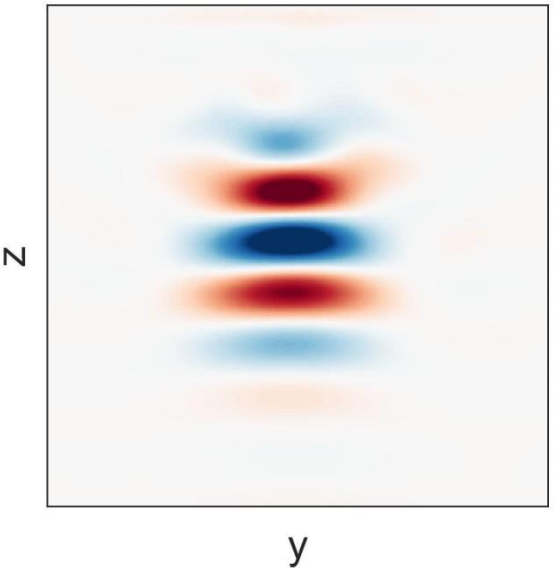
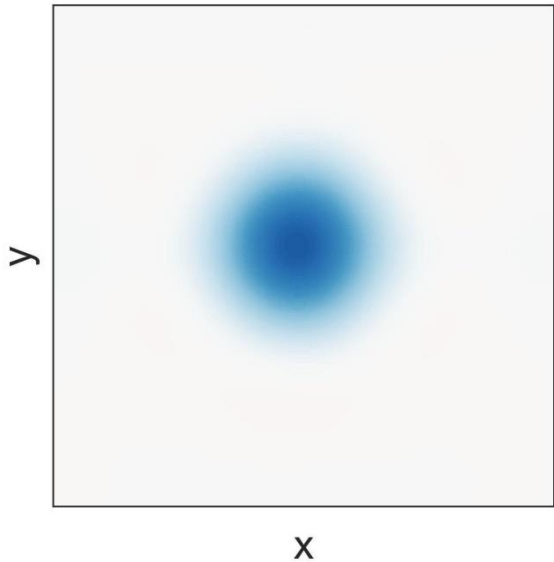
$$H_{\alpha\alpha}(\tilde{\mathbf{x}})$$



Multi-parameter Point Spread Function in Anisotropic Media



$$H_{\beta\alpha}(\tilde{\mathbf{x}})$$



Conclusions

1. Parameter resolution analysis based on scattering patterns should be improved;
2. Multi-parameter Hessian provides more complete measurements of parameter trade-offs;
3. Spike and stochastic probing methods can be used to quantify the parameter resolution and uncertainties in multi-parameter FWI;
4. In elastic FWI, S-wave velocity suffers from limited parameter crosstalk from P-wave and density;
5. Strong trade-off from S-wave velocity make P-wave and density difficult to be inverted;
6. The travel time contribution may result in the difficulty of inverting density;

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- Tiger cluster at Princeton University**
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- NSERC**
- SEG/Chevron Scholarship**
- Eyes High International Doctoral Scholarship**

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